INFINITE SETS OF POLYNOMIAL CONSERVED DENSITIES FOR NONLINEAR EVOLUTION EQUATIONS

A thesis
submitted in partial fulfilment
of the requirements for the Degree
of
Doctor of Philosophy in Physics
in the
University of Canterbury

by

Mark J. McGuinness

University of Canterbury
1978

ESSE NON VIDERI

CONTENTS

LIST OF FIGURES

ABSTRACT

The infinite sets of polynomial conserved densities which have been found for the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the Sine-Gordon equation, and the classical nonlinear shallow-water equations, are investigated using Noether's theorem. These sets are *identified* as energy or momentum densities of sets of higher-order integro-differential equations. These higher-order equations are obtained by operating n times on the evolution equation under consideration, with a nonlinear integro-differential operator, and hence their solution sets contain that of the evolution equation under consideration. The technique has possibilities for *predicting* the existence of infinite sets of polynomial conserved densities for other nonlinear evolution equations.

CHAPTER I

INTRODUCTION

Conservation laws have always been useful in helping to understand and solve the equations governing some system, and have played a part in the progress made in recent years in the attempt to solve certain *nonlinear* evolution equations. These nonlinear equations have possessed some remarkable features, including the existence of soliton solutions, which are solitary wave solutions whose shape and velocity are preserved after a collision with each other, the existence of a linear scattering technique for solving exactly the initial boundary-value problem, the existence of Bäcklund transformations mapping solutions to solutions, and the existence of an *infinite number of conservation laws*.

Since Noether's (1918) theorem, it has proved interesting and fruitful to relate the *conservation laws* of a system with the *symmetries* of that system. The symmetries of a system, in classical Lagrangian theory, are continuous groups of transformations leaving the action integral invariant. Noether's theorem associates these transformations with conservation laws of the system. By using Noether's theorem, it is possible to *identify* conservation laws by their associated transformation, and to *predict* conservation laws from the existence of the symmetry.

The existence of an infinite number of conservation laws for nonlinear systems is an unexpected and not very well understood phenomenon. In an attempt to improve upon this situation, this thesis will investigate such sets of conservation laws for several nonlinear evolution equations, using Noether's theorem. The conservation laws investigated will be *identified* as being equivalent to energy or momentum conservation laws for higher-order nonlinear equations, whose solution sets contain that of the nonlinear evolution equation under consideration. In the concluding chapter, a general technique with possibilities for *predicting* such infinite sets of conservation laws for other nonlinear equations will be presented.

Infinite sets of polynomial conserved densities [polynomials in the field variables and their derivatives] are easily derived for *linear* evolution equations. For example, the classical linear wave equation

$$\phi_{tt} - \phi_{xx} = 0 \tag{1.0.1}$$

is derivable from the Lagrangian

$$L = - \tfrac{1}{2}\phi_t\phi_t + \tfrac{1}{2}\phi_x\phi_x \quad . \tag{1.0.2}$$

Differentiating equation (1.0.1) n times with respect to x gives

$$(\phi_n)_{tt} - (\phi_n)_{xx} = 0 \quad . \tag{1.0.3}$$

Substituting

$$\psi \equiv \phi_n \tag{1.0.4}$$

in equation (1.0.3) yields

$$\psi_{tt} - \psi_{xx} = 0 \quad , \tag{1.0.5}$$

with the Lagrangian

$$L = -\tfrac{1}{2}\psi_t\psi_t + \tfrac{1}{2}\psi_x\psi_x \quad . \tag{1.0.6}$$

Since the Lagrangians (1.0.6) have no explicit dependence on time or space variables, Noether's theorem implies that energy and momentum are conserved for equations (1.0.5) for any n. Since solutions to equation (1.0.1) must also be solutions to equations (1.0.5), these energies and momenta constitute an infinite set of polynomial conserved densities for equation (1.0.1).

From an alternative point of view, using Rosen's generalised Noether's theorem, an equivalent set of energies and momenta may be obtained by differentiating equation (1.0.1) $2n$ times with respect to x, and multipling by ϕ_t [for energy], or ϕ_x [for momentum].

This approach can be generalised to vector field variables and several dimensions. One notable example is the derivation by Steudel (1965), who gives a generalised version of the preceding argument, and shows that such an argument explains Kibble's (1965) infinite number of conservation of *zilch* equations for electromagnetic fields.

If a wave equation is a continuous representation of a large number of discrete coupled systems with a large number of degrees of freedom, and hence a large number of integrals of the motion, it is not surprising that such an equation should possess an infinite number of conservation laws. This view applies whether the discrete system is linear or nonlinear, and hence there is no reason to believe that nonlinearity will prevent continuous wave equations from possessing infinite numbers of conservation laws. What is *not* clear is whether such laws will yield conserved densities in tractable form.

The above approach, of operating n times with a differential operator, fails when applied to *nonlinear* equations, as equations of the same form [such as equation (1.0.5)] do not follow from a nonlinear equation. The thrust of this investigation has been to show that more sophisticated operators, acting repeatedly on the nonlinear evolution equations under consideration, yield higher-order equations with polynomial conserved energy or momentum densities. Such operators have been found for several nonlinear wave equations of current interest, giving some insight into the form of these operators, which have been nonlinear and integro-differential in nature. It has not been possible to obtain general principles for the formation of such operators for arbitrary nonlinear wave equations, although general properties of these operators are pointed out.

Noether's theorem is reviewed in depth in chapter 2, followed by a brief review of the remarkable features mentioned above, for the case of the Korteweg-de Vries [KdV] equation, in sections (3.1) to (3.5). The KdV equation has been chosen because of the major part it has played in the discovery of these properties.

Summaries of some of Steudel's work are to be found in sections (3.6) and (4.2). Steudel has successfully used Noether's theorem to relate the infinite sets of conservation laws of the KdV and the modified KdV equations to infinitesimal extended Bäcklund transformations. The remainder of this thesis is original material.

Sections (3.7) and (3.9) identify the infinite set of polynomial conserved densities obtained by Gardner, Greene, Kruskal and Miura (1974) for the KdV equation as *energy* or *momentum* densities of higher-order integro-differential enveloping KdV equations, using Noether's theorem. Section (3.8) presents a corollary for the *generalised* KdV equations. An application to the linearised KdV equation is given in section (3.10).

In chapter 4, Noether's theorem is used to identify the infinite set of polynomial conserved densities for the modified KdV equation as equivalent to *energy* or *momentum* densities of higher-order modified KdV equations, whose solution sets contain that of the modified KdV equation.

The Sine-Gordon equation, which also possesses an infinite set of polynomial conserved densities, is shown in chapter 5 to yield to the same approach used in chapters 3 and 4, but with some differences in results. In particular, an infinite set of momentum densities is derived for the higher-order Sine-Gordon equations, and the energy densities of these equations are shown to be zero.

The technique of previous chapters is extended to a *matrix* formalism in chapter 6, and is used to identify an infinite set of polynomial conserved densities for the classical nonlinear shallow-water equations as energy or momentum densities of higher-order equations.

An overview of the technique used in this thesis is presented in chapter 7. In section (7.1), the corresponding results for linear equations are derived, and comments on the possibilities for generalising the technique may be found in section (7.2).

CHAPTER II

NOETHER'S THEOREM

Noether's theorem [Noether, (1918)] provides a natural way of associating a conserved quantity for a system with an infinitesimal transformation on that system. This association can serve as a means of identifying the conserved quantity. For example, conservation of energy or momentum is associated with invariance of the system under an infinitesimal time or space translation, conservation of angular momentum is associated with invariance under an infinitesimal rotation, and conservation of charge is associated with invariance under an infinitesimal phase or gauge transformation of the field variables. Noether's theorem also provides a general framework for predicting which systems or equations will have a certain type of conservation law. For example, if the Lagrangian density for a system has no explicit time-dependence, Noether's theorem says that energy is conserved in that system [see appendix C].

A brief statement of the theorem is presented here. Define the *action integral*

$$J \equiv \int_v L \, dx \tag{2.0.1}$$

over a volume v in four-space, where L is the Lagrangian density for a system. The Lagrangian density is a function in terms of which the equation of motion for a system may be expressed, as will be seen in section (2.2).

Noether's first theorem says that *if the action integral J is invariant*[1] *under the infinitesimal one-parameter transformation*

$$x' = x + \delta x(x) \tag{2.0.2}$$

$$\phi'(x) = \phi(x) + \bar{\delta}\phi(x) \; ,$$

say

$$\delta J = \int_v (-d_\mu \; G^\mu) dx \; , \tag{2.0.3}$$

where G^μ *is zero on the boundary of volume* v *so that* δJ *is zero, then the following relation* [Noether's relation] *holds:*

$$\boxed{-\bar{\delta}\phi \; E_\phi(L) = d_\mu[\pi^\mu(L) \; \bar{\delta}\phi + L\delta x^\mu + G^\mu] \; ,} \tag{2.0.4}$$

where

$$E_\phi(L) \equiv \sum_{a=1} (-1)^a \; d_{\mu_1} \cdots d_{\mu_a} \left(\frac{\delta L}{\delta\phi_{\mu_1 \cdots \mu_a}}\right) \; , \tag{2.0.5}$$

$$\pi^\mu(L)\bar{\delta}\phi \equiv \sum_{a=0} \sum_{b=0}^{a} (-1)^b \; d_{\mu_1} \cdots d_{\mu_b} \left(\frac{\partial L}{\partial\phi_{\mu\mu_1\cdots\mu_a}}\right) d_{\mu_{b+1}} \cdots d_{\mu_a} \bar{\delta}\phi \; , \tag{2.0.6}$$

and

1 Within a divergence, for all arbitrary volumes of integration

$$\phi_\nu \equiv d_\nu\phi \equiv \frac{\partial\phi(x)}{\partial x^\nu} \quad ,$$

$$d_\nu[F(x,\phi,\phi_\mu)] = \frac{\partial F}{\partial x^\nu} + \frac{\partial F}{\partial \phi}\phi_\nu + \frac{\partial F}{\partial \phi_\mu}\phi_{\mu\nu} \quad ,$$

$$x \equiv \{x^0,x^1,x^2,x^3\} \equiv \{t,x,y,z\} \quad ,$$

$$dx \equiv dx^0dx^1dx^2dx^3 \quad .$$

The operator E is known as the Euler-Lagrange operator. Note that repeated indices are summed upon and can be treated as dummy indices, and that d_ν is a total derivative whereas ∂_ν is a partial derivative, regarding $\phi(x)$ as independent of x^ν. That is, ∂_ν is a derivative with respect to *explicit* x^ν - dependence [not via ϕ].

Noether's relation (2.0.4) associates the variation $(\bar{\delta}\phi,\delta x)$ with the conservation equation

$$\boxed{d_\mu[\pi_\mu(L)\bar{\delta}\phi + L\delta x^\mu + G^\mu] = 0 \quad .} \tag{2.0.7}$$

This is because, as will be seen in section (2.2),

$$E_\phi(L) = 0 \tag{2.0.8}$$

is the equation of motion for the system with action integral J.

The variation $(\bar{\delta}\phi,\delta x)$ is an *invariance transformation* in the sense that J is functionally invariant under it.

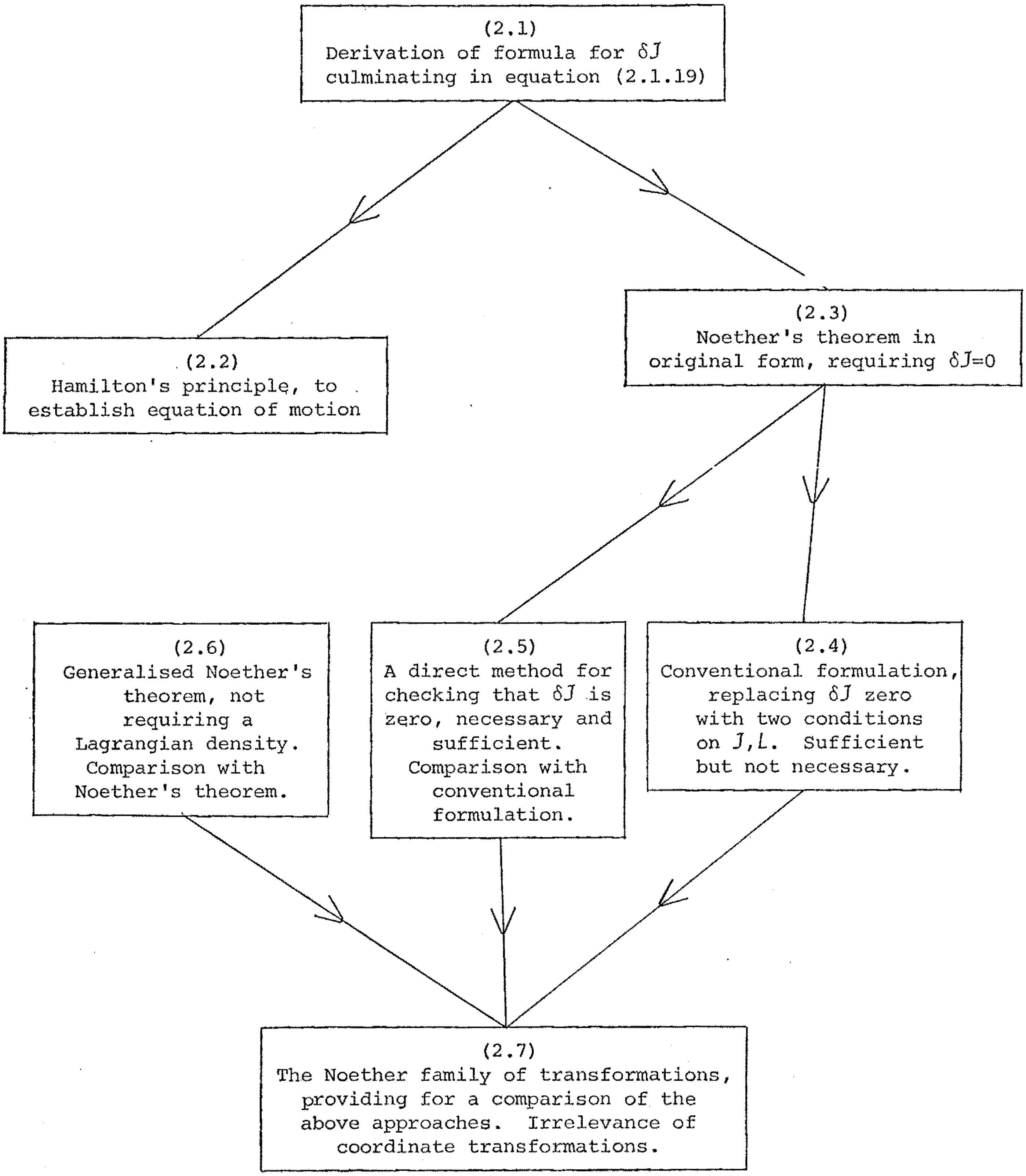
Since Noether's 1918 paper, there have been many by other authors, interpreting, extending and applying her's theorem in physics. Rosen (1971) gives a icularly good account of conventional formulations he theorem; Hill (1951) has a clear derivation; oder (1968) gives a clear statement of the original theorem; Trautman (1967) has a mathematically rigorous treatment; Boyer (1966, 1967), Dass (1966) and Dothan (1972) also have interesting approaches to the theorem.

These differing approacl l interpretations can be confusing to the uninitiated. This chapter attempts to present a unified account of these approaches, discussing their advantages and disadvantages.

The organisation of this chapter is presented in flow chart form in figure (2.1), to help clarify the development. In section (2.1), a mathematical formula will be derived for the functional variation of the action integral J under the infinitesimal transformation $(\bar{\delta}\phi, \delta x)$. This formula is fundamental to the proof of Noether's theorem. A brief excursion is made into Hamilton's principle in section (2.2), to establish the equations of motion for a system. Noether's theorem is proved in section (2.3) using the formula for δJ from section (2.1), and the existence of a conserved quantity is shown to arise from the theorem. This form of Noether's theorem, close in spirit to the original paper, is a largely mathematical one, with the requirement for a conservation law that δJ be zero. One popular interpretation of this requirement is found in section (2.4), in which the requirement is replaced by two separate requirements, each of which has an obvious

FIGURE (2.1)

A Schematic of the Development of Chapter II



physical implication. However, these requirements are sufficient but not necessary for δJ to be zero. An alternative approach, sufficient and necessary for δJ to be zero, is presented in section (2.5). It is shown that this approach yields a more direct method for checking that δJ is zero, than that in section (2.4). In section (2.6) a quite different approach to conservation laws is presented. Called *generalised Noether's theorem*, this approach does not explicitly require a Lagrangian density, but does require the equation of motion. This can be of some advantage as some equations do not have Lagrangian densities. This approach uses a relation similar to Noether's relation (2.0.4) as a starting point, rather than using the expression derived for δJ in section (2.1). Generalised Noether's theorem is shown to be mathematically equivalent to Noether's theorem if a Lagrangian density exists for the system. In section (2.7), the concept of a Noether family of transformations $(\delta\phi, \delta x)$ is introduced, facilitating a comparison of the three approaches to Noether's theorem.

(2.1) THE VARIATION OF THE ACTION INTEGRAL

The derivation of Noether's theorem involves the determination of the functional variation of the action integral

$$J = \int_V L \, dx \quad , \qquad (2.1.1)$$

where the Lagrangian density

$$L = L(x,\phi,\phi_{\mu},\phi_{\mu\nu} \ldots) \tag{2.1.2}$$

is a function of the independent variables [coordinates] x^{μ} and of the dependent or field variable ϕ and its derivatives to any order. This function characterises any system described by an Euler-Lagrange equation of motion, as is shown in section (2.2). A system may have several Lagrangians and several field variables, and the field variables may be tensor quantities. For clarity and ease of notation this derivation will deal with one Lagrangian density depending on one scalar field variable and any number of derivatives of that variable. The generalisation to several Lagrangians, several field variables and tensor quantities is easily made.

The functional variation of the action integral under the infinitesimal transformation

$$\delta x \equiv x' - x = \delta x(x) \tag{2.1.3}$$

$$\delta\phi \equiv \phi'(x') - \phi(x)$$

is defined as

$$\delta J \equiv \int_{V'} L(x',\phi'(x'), \ldots)dx' - \int_{V} L(x,\phi(x), \ldots)dx \tag{2.1.4}$$

where v' is the transformation of the four-volume v under δx. δx will be treated as a function of x only, although it can be generalised to a function of x and ϕ and its derivatives [e.g., Dothan (1972)].

J will be expressed in the form of a four-divergence term and a remainder term, using the infinitesimal nature of the transformation, and retaining only terms that are up to first-order in the transformation.

To first-order in the variation,

$$dx' = dx'dy'dz'dt' = [1 + d_\mu(\delta x^\mu)]dx \tag{2.1.5}$$

so that

$$\delta J = \int_v \{L[x'] - L[x] + L[x']\; d_\mu(\delta x^\mu)\}dx \tag{2.1.6}$$

where

$$L[x'] \equiv L(x', \phi'(x'), \ldots) \quad .$$

Using a Taylor expansion,

$$L[x'] - L[x] = \frac{\partial L}{\partial x^\mu}\,\delta x^\mu + \sum_a \frac{\partial L}{\partial \phi_{\mu_1 \cdots \mu_a}}\,\delta\phi_{\mu_1 \cdots \mu_a} \;, \tag{2.1.7}$$

where the sum is over *combinations* of $\{\mu_1, \cdots, \mu_a\}$.

To first order in the variation,

$$L[x']\; d_\mu(\delta x^\mu) = L[x]\; d_\mu(\delta x^\mu) \quad , \tag{2.1.8}$$

so that using equations (2.1.7) and (2.1.8), equation (2.1.6) becomes

$$\delta J = \int_V [\sum_a \frac{\partial L}{\partial \phi_{\mu_1 \cdots \mu_a}} \delta\phi_{\mu_1 \cdots \mu_a} + \frac{\partial L}{\partial x^\mu} \delta x^\mu + L d_\mu (\delta x^\mu)] dx \tag{2.1.9}$$

It is useful here to define a different field variation to that used so far,

$$\bar{\delta}\phi(x) \equiv \phi'(x) - \phi(x) \quad . \tag{2.1.10}$$

The definitions of the field variations can be used to show that

$$\bar{\delta}\phi = \delta\phi - \delta x^\mu \phi_\mu \quad , \tag{2.1.11}$$

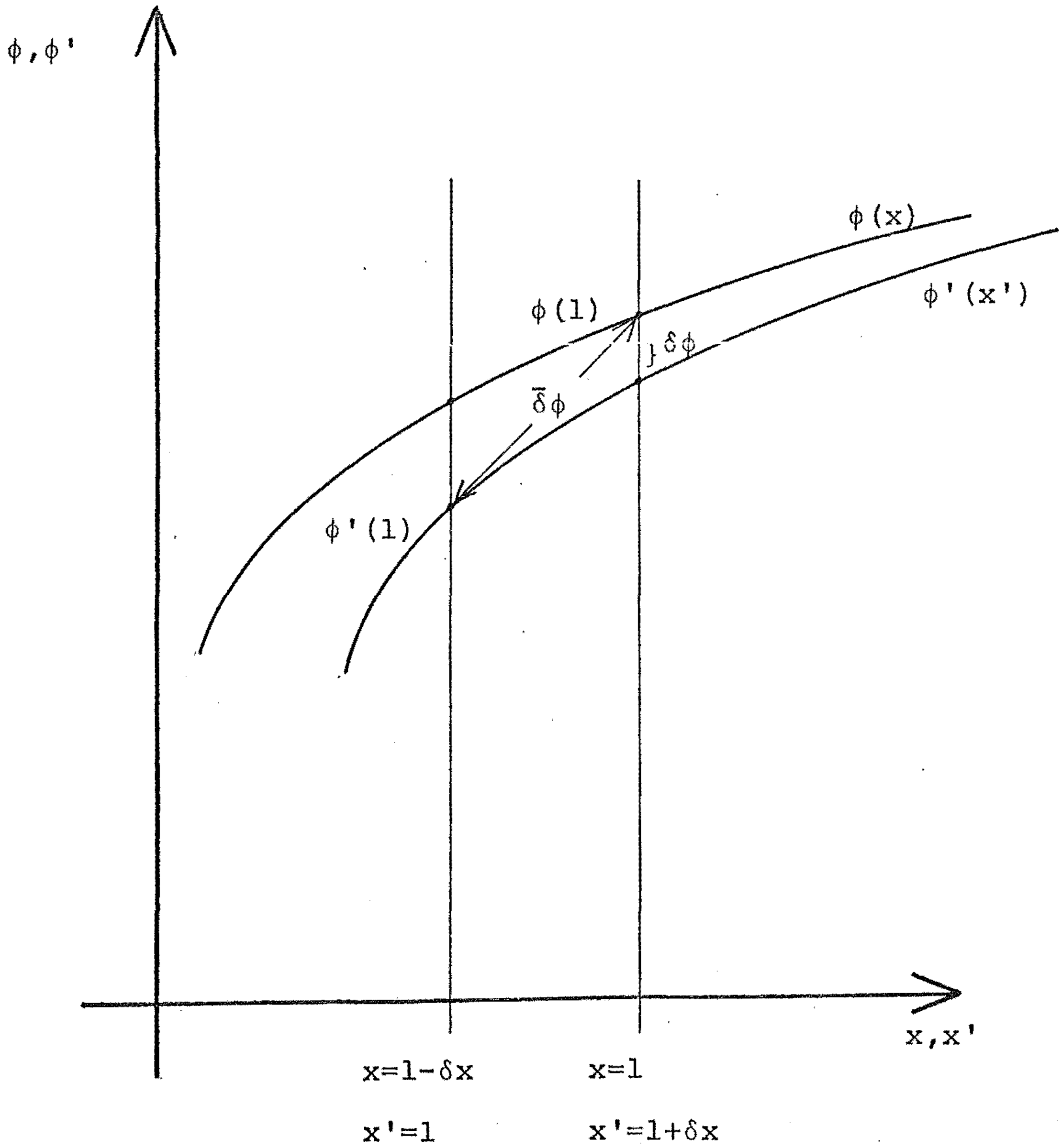
to first order in the variations. Hence, either $(\delta\phi, \delta x)$ or $(\bar{\delta}\phi, \delta x)$ is sufficient to define the variation or transformation used.

The first field variation $\phi'(x') - \phi(x)$ can be viewed as the change in the field variable *at a fixed point in space*, that point having the coordinates (x) in the untransformed system and (x') in the transformed system. In this sense, $\delta\phi$ is the true change in the field ϕ at a point.

The second field variation $\phi'(x) - \phi(x)$ is the change in ϕ *at a fixed grid reference point or set of coordinates* (x). This grid reference point is not necessarily

FIGURE (2.2)

The Variations $\delta\phi$, $\bar{\delta}\phi$, for a Transformation $(\delta x, \delta\phi)$



fixed in space under the transformation, since the transformation may explicitly change the coordinate system. See figure (2.2) for a pictorial representation of $\delta\phi$, $\bar{\delta}\phi$ for the case of a variation δx in a single x - coordinate and a field variation $\delta\phi$ which is nonzero.

The advantage of using $\bar{\delta}\phi$ instead of $\delta\phi$ is that $\bar{\delta}$ commutes with differentiation, that is

$$d_\mu(\bar{\delta}\phi) = \bar{\delta}(\phi_\mu) = \bar{\delta}\phi_\mu \quad , \tag{2.1.12}$$

whereas, realising that the definitions for $\delta\phi$ and $\bar{\delta}\phi$ also apply when ϕ is replaced by $\phi_{\mu_1 \ldots \mu_a}$,

$$\begin{aligned} d_\mu(\delta\phi) &= d_\mu[\phi'(x') - \phi(x)] \\ &= \frac{d}{dx^{\nu'}}[\phi'(x')]\frac{dx^{\nu'}}{dx^\mu} - \phi_\mu(x) \\ &= \phi'_\nu(x')[\delta^\mu_\nu + d_\mu(\delta x^\nu)] - \phi_u(x) \\ &= \delta(\phi_\mu) + \phi_\nu\, d_\mu(\delta x^\nu) \quad , \end{aligned} \tag{2.1.13}$$

to first order in δx^μ.

Realising that ϕ can be replaced by $\phi_{\mu_1 \ldots \mu_a}$ in equation (2.1.11), equation (2.1.9) becomes

$$\delta J = \int_V [\sum_a \frac{\partial L}{\partial\phi_{\mu_1 \ldots \mu_a}} \bar{\delta}\phi_{\mu_1 \ldots \mu_a} + \sum_a \frac{\partial L}{\partial\phi_{\mu_1 \ldots \mu_a}} \phi_{\mu_1 \ldots \mu_a \nu}\, \delta x^\nu +$$

$$+ \frac{\partial L}{\partial x^\mu} \delta x^\mu + L d_\mu (\delta x^\mu)]dx \ , \tag{2.1.14}$$

$$= \int_V [\sum_a \frac{\partial L}{\partial \phi_{\mu_1} \cdots \mu_a} \bar{\delta}\phi_{\mu_1} \cdots \mu_a + d_\mu (L\delta x^\mu)]dx. \tag{2.1.15}$$

Using the commutative property (2.1.12), it can be shown that

$$\sum_a \frac{\partial L}{\partial \phi_{\mu_1} \cdots \mu_a} \bar{\delta}\phi_{\mu_1} \cdots \mu_a = d_\mu [\pi^\mu (L) \bar{\delta}\phi] + \bar{\delta}\phi E_\phi (L) \ , \tag{2.1.16}$$

where it will be recalled that

$$E_\phi (L) \equiv \sum_{a=0} (-1)^a \, d_{\mu_1} \cdots d_{\mu_a} \left(\frac{\partial L}{\partial \phi_{\mu_1} \cdots \mu_a} \right) \tag{2.1.17}$$

$$\pi^\mu (L) \bar{\delta}\phi \equiv \sum_{a=0} \sum_{b=0} (-1)^b \, d_{\mu_1} \cdots d_{\mu_b} \left(\frac{\partial L}{\partial \phi_{\mu\mu_1} \cdots \mu_a} \right) d_{\mu_{b+1}} \cdots d_{\mu_a} \, \delta\phi \ , \tag{2.1.18}$$

[see appendix A]. Substituting this result into equation (2.1.15) gives the desired form for the functional variation of the action integral:

$$\boxed{\delta J = \int_V \{d_\mu [\pi^\mu (L) \bar{\delta}\phi + L\delta x^\mu] + \bar{\delta}\phi E_\phi (L)\}dx \quad .} \tag{2.1.19}$$

Equation (2.1.19) is the key equation for the derivation of the Euler-Lagrange equation of motion from Hamilton's principle, and for the proof of Noether's theorem.

(2.2) HAMILTON'S PRINCIPLE AND THE EULER-LAGRANGE EQUATION OF MOTION

The generalisation of Hamilton's principle to field theory can be expressed as [Saletan and Cromer, (1971)]:

The true physical field variables are those with the [arbitrary] required boundary values for which the action integral J is an extremum.

In other words, for a field variation

$$\phi' = \phi + \delta\phi = \phi + \bar{\delta}\phi$$

with the restriction that $\delta\phi$ be zero on the boundary of integration in J, Hamilton's principle says that for solutions, δJ must be zero. The restriction on $\delta\phi$ ensures that the boundary values of ϕ are fixed.

If δJ is zero, equation (2.1.19) becomes

$$\oint_B [\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu]d\sigma^\mu + \int_V \bar{\delta}\phi E_\phi(L) = 0 \ , \tag{2.2.1}$$

using the divergence theorem, where B is the boundary of v and $d\sigma^\mu$ is an element on that surface. Hamilton's principle requires that δx be zero everywhere and $\delta\phi$ be zero on B, so that equation (2.2.1) becomes

$$\int_V \bar{\delta}\phi E_\phi(L) = 0 \quad . \tag{2.2.2}$$

Since $\bar{\delta}\phi$ is arbitrary everywhere inside v, this implies that

$$\boxed{E_{\phi}(L) = 0} \tag{2.2.3}$$

for solutions of the system. This is the Euler-Lagrange equation of motion for the system.

For the Lagrangian density

$$L = L(\phi, \phi_{\mu}) \tag{2.2.4}$$

for example, the Euler-Lagrange equation of motion is

$$-\frac{d}{dx^{\mu}}\left(\frac{\partial L}{\partial \phi_{\mu}}\right) + \frac{\partial L}{\partial \phi} = 0 \quad . \tag{2.2.5}$$

This derivation is reversible, so that Hamilton's principle is obeyed if and only if the Euler-Lagrange equation holds. The generalization to systems with several equations of motion and with tensor field variables is easily made.

In practice, the equation of motion for a system is often known, and is cast into the Euler-Lagrange form to obtain a Lagrangian density for that system.

(2.3) NOETHER'S THEOREM

Noether's first theorem will now be proved. If δJ is zero[1], equation (2.1.19) becomes

$$\int_V \{d_{\mu}[\pi^{\mu}(L)\bar{\delta}\phi + L\delta x^{\mu}] + \bar{\delta}\phi E_{\phi}(L)\}dx = 0 \quad . \tag{2.3.1}$$

1 see footnote, p.9.

This is true if and only if the integrand is

$$d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu] + \bar{\delta}\phi E_\phi(L) = - d_\mu G^\mu \qquad (2.3.2)$$

where G^μ is zero on the boundary B of V, since the divergence theorem then gives

$$\delta J = \int_V (-d_\mu G^\mu)\,dx = \oint_B G^\mu . d\sigma^\mu = 0 \quad . \qquad (2.3.3)$$

It should be noted that the boundary of v is usually chosen such that ϕ and its derivatives are zero everywhere on it. Then any function $G^\mu(\phi, \phi_\mu, \ldots)$ will be zero on that boundary.

Clearly, equation (2.3.2) can be rearranged to give *Noether's relation:*

$$\boxed{-\bar{\delta}\phi E_\phi(L) = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu + G^\mu],} \qquad (2.3.4)$$

and Noether's theorem is proved, as stated at the beginning of this chapter.

The connection between Noether's relation and the existence of a conserved quantity will now be established. Recall the equation of motion for a system from section (2.2),

$$E_\phi(L) = 0 \quad . \qquad (2.3.5)$$

Hence, if Noether's relation holds,

$$\boxed{d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu + G^\mu] = 0} \tag{2.3.6}$$

for solutions to the equation of motion. An equation of this form, equating a four-divergence to zero, is known as a *conservation equation* or a *conservation law*. This is because, under an assumption which is quite acceptable physically, such an equation gives rise to a conserved quantity, that is, something which does not change with time.

To see this, a distinction is made between time and space coordinates, so that equation (2.3.6) is written as

$$d_t[\pi^t(L)\bar{\delta}\phi + L\delta t + G^t] + \frac{d}{dx^a}[\pi^a(L)\bar{\delta}\phi + L\delta x^a + G^a] = 0 \; , \tag{2.3.7}$$

where

$$x^a = x,y,z \quad \text{and} \quad a = 1,2,3 \; .$$

Integrating this equation over a three-space volume R and using the divergence theorem, it becomes

$$d_t\int_R [\pi^t(L)\bar{\delta}\phi + L\delta t + G^t]d\overset{3}{x} + \oint_S [\pi^a(L)\bar{\delta}\phi + L\delta x^a + G^a]\, d\sigma^a = 0 \; . \tag{2.3.8}$$

The assumption is made here that R can be chosen such that ϕ and its derivatives vanish on its boundary S. One interpretation of this assumption is that R is chosen to completely contain the system. One common choice of R is "all space", with boundaries at $\pm\infty$. The assumption then is that for any physical system, the field variable and its derivatives vanish if one goes far enough away from the system.

Under this assumption, the surface integral in equation (2.3.8) vanishes, and

$$d_t \int_R [\pi^t(L)\bar{\delta}\phi + L\delta t + G^t]d^3x = 0 \qquad (2.3.9)$$

for solutions. Hence the integral in equation (2.3.9) is a conserved quantity for the system, and

$$\pi^t(L)\bar{\delta}\phi + L\delta t + G^t \qquad (2.3.10)$$

is a conserved density for the system.

In this way, Noether's theorem relates a transformation $(\delta x, \bar{\delta}\phi)$ *leaving the action integral functionally invariant*, to a conserved quantity for the system. Note that the conserved quantity may turn out to be trivial, e.g. identically zero. For a discussion and classification of types of conserved densities, see appendix D.

(2.4) CONVENTIONAL NOETHER'S THEOREM

Noether's theorem in the previous section requires that δJ be zero for an infinitesimal transformation to yield a conservation law. The problem is how to check that this condition is met, for a particular transformation and a particular system with Lagrangian L.

Many conventional formulations of Noether's theorem [e.g. Hill (1951), Boyer (1965), Dothan (1972)] check that δJ is zero by making two separate requirements:

(A) *It is required that L transform as a scalar density*, that is

$$\boxed{L'[x']dx' = L[x]dx \quad ,} \tag{2.4.1}$$

where

$$dx \equiv dxdydzdt$$

and where L' is defined as the transformed Lagrangian density such that the equation of motion in the transformed system is

$$E_{\phi'}[L'] = 0 \quad . \tag{2.4.2}$$

This requirement (2.4.1) is itself treated as a *definition* of L', although it is strictly an *assumption* about the system being transformed.

Note that this requirement can be equivalently written

$$\int_{v'} L'[x']dx' - \int_{v} L[x]dx = 0 \tag{2.4.3}$$

for all volumes of integration v, that is, by definition,

$$\delta|J| = 0 \quad , \tag{2.4.4}$$

the *numerical* variation of the action integral is zero for all volumes of integration.

(B) *It is also required that under the transformation,*

$$\boxed{\bar{\delta} L = d_\mu G^\mu(x,\phi,\phi_\mu \cdots)} \quad , \tag{2.4.5}$$

where

$$\bar{\delta} L \equiv L'[x] - L[x] \quad , \tag{2.4.6}$$

and G^μ is zero on the boundary of integration of J. That is, *L is required to be form-invariant within a divergence term.* This ensures form-invariance of the equation of motion, as will be discussed later in this section.

So the requirement that δJ be zero has been replaced in these conventional formulations of Noether's theorem by the two separate requirements (A) and (B), both with clear physical interpretations on them. *These two requirements are sufficient to ensure that δJ is zero, but they are hardly necessary* [see Rosen (1971) for a detailed discussion of this point]:

To see that requirements (2.4.1) and (2.4.4) are sufficient to ensure δJ is zero, recall that

$$\delta J \equiv \int_{V'} L[x']dx' - \int_V L[x]dx \tag{2.4.7}$$

Using assumption (2.4.1), this becomes

$$\delta J = \int_{V'} \{L[x'] - L'[x']\}dx' \quad . \tag{2.4.8}$$

Using the definition (2.4.5) of $\bar{\delta}L$ in terms of the primed variables, this becomes

$$\delta J = \int_{V'} \{- \bar{\delta}L[x']\}dx' \quad . \tag{2.4.9}$$

Since $\bar{\delta}$ and dx' are both infinitesimal quantities, using a Taylor expansion gives to first order in the variation:

$$\bar{\delta}L[x']dx' = \bar{\delta}L[x]dx \quad , \tag{2.4.10}$$

so that equation (2.4.9) becomes

$$\delta J = - \int_{V} \bar{\delta}L \; dx \quad . \tag{2.4.11}$$

Substituting requirement (2.4.4) into this gives

$$\delta J = - \int_{V} (d_\mu G^\mu) dx \quad , \tag{2.4.12}$$

which by the reasoning of equation (2.3.3) gives δJ as zero.

To see that the requirements (A) and (B) of conventional Noether's theorem are not necessary for δJ to be zero, take the particular case of a transformation such that

$$L'[x']dx' = L[x]dx + K(x,\phi,\phi_\mu, \cdots)dx \tag{2.4.13}$$

and

$$\bar{\delta} L = d_\mu G^\mu + K(x,\phi,\phi_\mu, \ldots) \quad . \tag{2.4.14}$$

Clearly, conditions (A) and (B) are not satisfied, yet when equations (2.4.13) and (2.4.14) are substituted into the definition (2.4.7) of δJ, in the same manner as then, the function K cancels out, leaving

$$\delta J = \int_V (-d_\mu G^\mu)\,dx \quad , \tag{2.4.15}$$

that is, δJ is zero as in equation (2.3.3).

Hence nonfulfillment of the requirements (A) and (B) can lead to an invariant action integral [and hence to a conservation law], so that these two assumptions of conventional Noether's theorem are sufficient but not necessary that δJ be zero.

(2.4.i) Symmetry and Invariance Transformations

The strict definition of a symmetry transformation [Trautman (1967)] is *any map which carries solutions of the equation of motion to solutions of the same equation*. Hence, under a symmetry transformation, the equation of motion may change in form, as long as the solution set is unchanged. An invariance transformation is defined as *one which leaves the equation of motion form-invariant*. Some authors [e.g. Hill (1951)] mistakenly define a symmetry transformation as an invariance transformation is defined here, which can lead to some confusion [see

Boyer (1967) for a discussion of symmetry transformations]. After making such a definition, it is asserted that a symmetry transformation leads to a conservation law via Noether's theorem. Such an assertion must be viewed with care, realising exactly what is meant by "symmetry transformation". Usually it is meant that the transformation satisfies requirements (A) and (B). According to the correct definition given above, requirement (B) does ensure a symmetry transformation, but a *symmetry transformation does not necessarily ensure that requirements (A) or (B) are met, and does not necessarily lead to a conservation law.*

An example of a symmetry transformation which does not lead to a conservation law is one such that

$$\bar{\delta} L = \varepsilon L, \quad \delta x = 0, \quad \delta\phi \neq 0. \tag{2.4.16}$$

The transformed equation of motion is

$$E_{\phi'}\{L'[x']\} = E_{\phi'}\{L[x'] + \bar{\delta} L[x']\}, \tag{2.4.17}$$

$$= E_{\phi'}\{L[x']\}(1 + \varepsilon) \ . \tag{2.4.18}$$

Clearly, the new equation of motion

$$E_{\phi'}\{L(x, \phi'(x), \phi'_{\mu}(x), \ \ldots \ )\} = 0 \tag{2.4.19}$$

has the same solution set $\{\phi'\}$ as the original equation,

$$E_\phi\{L(x,\phi(x),\phi_\mu(x),\ \ldots\ \} = 0 \quad . \tag{2.4.20}$$

Hence the transformation is strictly a symmetry transformation, yet $\bar{\delta}L$ is not a divergence for nontrivial L, so that requirement (B) is not met, and in general a conservation law does not follow.

To see that requirement (B) is equivalent to the requirement of an invariance transformation, and hence ensures a symmetry transformation, recall the transformed equation of motion

$$E_{\phi'}\{L'[x']\} = 0 \quad . \tag{2.4.21}$$

Using requirement (B) [equation (2.4.4)] in terms of the primed variables, the left-hand side of equation (2.4.21) becomes

$$E_{\phi'}\{L'[x']\} = E_{\phi'}\{L[x'] + \bar{\delta}L[x']\} \quad , \tag{2.4.22}$$

$$= E_{\phi'}\{L[x']\} + E_{\phi'}\{-d_\mu G^\mu[x']\} \quad . \tag{2.4.23}$$

Appendix E has a proof of the general identity that

$$E_\phi\{d_\mu G^\mu\} = 0 \tag{2.4.24}$$

for any $G(x,\phi,\phi_\mu\ \cdots)$. Using this, equation (2.4.23) becomes

$$E_{\phi'}\{L'[x']\} = E_{\phi'}\{L[x']\} \quad , \tag{2.4.25}$$

so that the transformed equation of motion is

$$E_{\phi'}\{L[x']\} = 0 \quad . \tag{2.4.26}$$

Clearly, this is of the same functional form as the original equation of motion

$$E_{\phi}\{L[x]\} = 0 \quad . \tag{2.4.27}$$

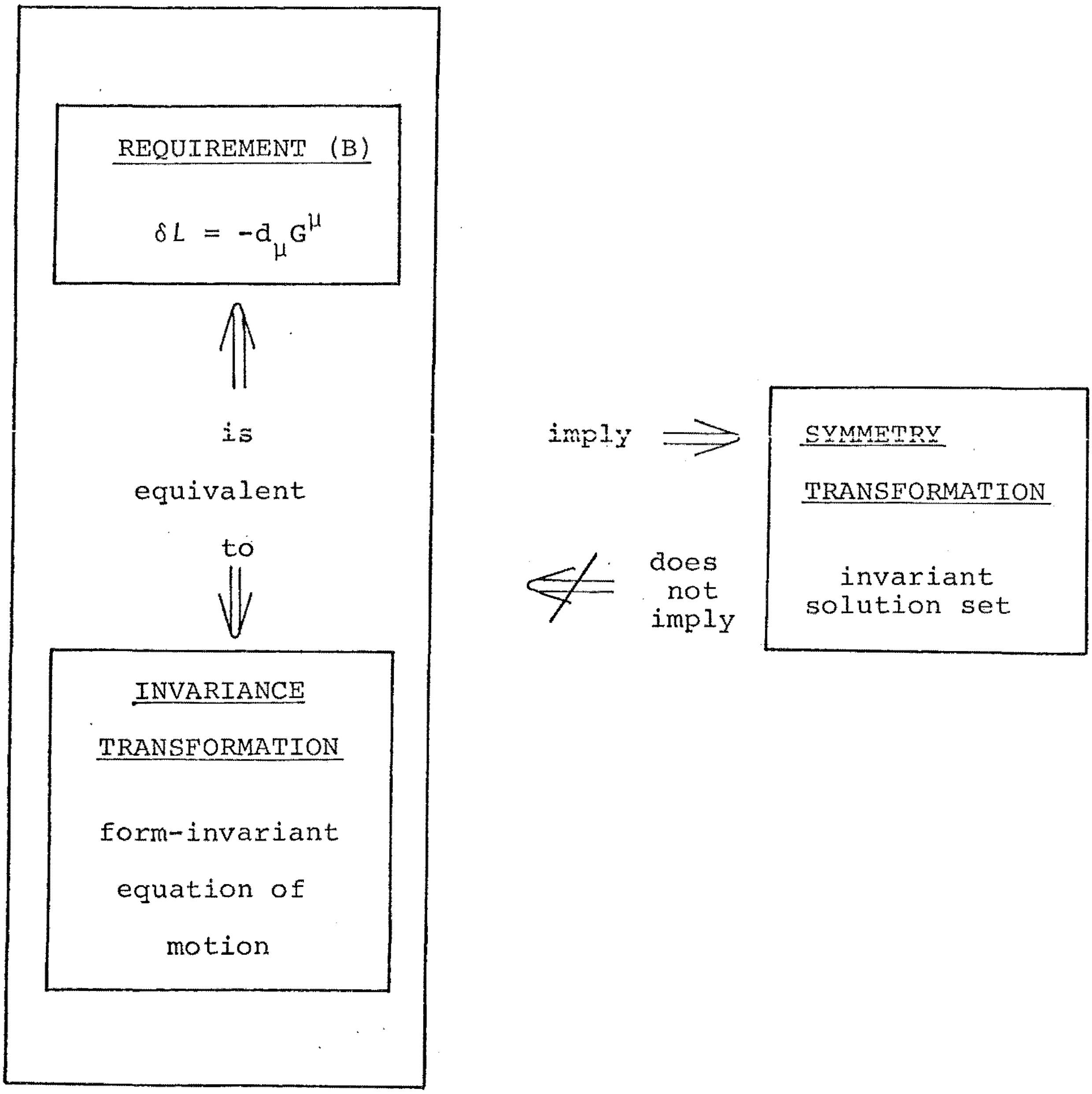
Hence the transformation is an invariance transformation, and since this process is reversible, requirement (B) is equivalent to the assumption of an invariance transformation. Since an invariance transformation must leave the solution set invariant, requirement (B) ensures the transformation is a symmetry transformation.

Refer to figure (2.3) for a schematic of the relationships between requirement (B), invariance and symmetry transformations.

Although conventional Noether's theorem is a more restrictive formulation than is necessary, it has enjoyed some success in application. One reason for this is that many physical systems [e.g. linear field systems, discrete many-body systems] have conservation laws associated with transformations under which requirements (A) and (B) are met. Another reason could be the ready physical interpretations of these requirements. For an example of

FIGURE (2.3)

A Schematic of the Relationships Between Requirement (B), Invariance Transformations, and Symmetry Transformations.



the use of conventional Noether's theorem on those systems with a scalar Lagrangian density, see appendix B.

A less restrictive method for checking that a transformation leaves the action integral functionally invariant is discussed in the next section.

(2.5) AN ALTERNATIVE APPROACH TO NOETHER'S THEOREM

The problem in applying Noether's theorem of section (2.3) is to check that δJ is zero. The technique in the previous section (2.4) has the drawback of being sufficient but not necessary that δJ be zero, and hence of being more restrictive than necessary. An approach is presented in this section that gives a necessary and sufficient condition that δJ be zero.

A necessary and sufficient condition that the functional variation of the action integral be zero is that

$$\boxed{\delta L = -d_\mu G^\mu(x,\phi,\phi_\mu \;\ldots\;)} \quad , \tag{2.5.1}$$

where

$$\delta L \equiv \sum_a \frac{\partial L}{\partial \phi_{\mu_1 \;\ldots\; \mu_a}} \delta\phi_{\mu_1 \;\ldots\; \mu_a} + \frac{\partial}{\partial x^\mu}(L\delta x^\mu) \quad , \tag{2.5.2}$$

and where G^μ *vanishes on the boundary of* v *and* J.

To prove this, recall equation (2.1.9) for the functional variation of J:

$$\delta J = \int_v \left[\sum_a \frac{\partial L}{\partial \phi_{\mu_1 \ldots \mu_a}} \delta \phi_{\mu_1 \ldots \mu_a} + \frac{\partial}{\partial x^\mu} (L \delta x^\mu) \right] dx \quad , \qquad (2.5.3)$$

which gives, from definition (2.5.2),

$$\delta J = \int_v \delta L dx \quad . \qquad (2.5.4)$$

Then clearly, δJ is zero if and only if δL is the divergence of a term that vanishes on the boundary of v, as discussed at equation (2.3.3).

The condition (2.5.1) is related to requirements (A) and (B) as follows:

Recall requirement (A) as expressed in equation (2.4.1),

$$L'[x']dx' = L[x]dx \quad . \qquad (2.5.5)$$

Using equation (2.1.5) for the transform of the infinitesimal four-volume dx, this becomes

$$\{L'[x'] - L[x] + L'[x']d_\mu(\delta x^\mu)\}dx = 0 \quad , \qquad (2.5.6)$$

or, to first-order in the variation, and since dx is nonzero,

$$L'[x'] - L[x] + L[x]d_\mu(\delta x^\mu) = 0 \quad . \qquad (2.5.7)$$

Defining

$$[\delta L] \equiv L'[x'] - L[x] + L[x]d_\mu(\delta x^\mu) \quad , \tag{2.5.8}$$

requirement (A) becomes, in another form,

$$\boxed{[\delta L] = 0} \quad . \tag{2.5.9}$$

Requirement (B) is recalled to be

$$\bar{\delta} L \equiv L'[x] - L[x] = d_\mu G^\mu \quad . \tag{2.5.10}$$

Using the definitions (2.5.8) and (2.5.10),

$$[\delta L] - \bar{\delta} L = L'[x'] + L[x]d_\mu(\delta x^\mu) - L'[x] . \tag{2.5.11}$$

A Taylor expansion gives

$$[\delta L] - \bar{\delta} L = \sum_a \frac{\partial L'}{\partial \phi'_{\mu_1 \cdots \mu_a}} \delta\phi'_{\mu_1 \cdots \mu_a} + \frac{\partial L'}{\partial x^\mu} \delta x^\mu + L[x]d_\mu(\delta x^\mu) \tag{2.5.12}$$

which is, to first order in the variation,

$$[\delta L] - \bar{\delta} L = \sum \frac{\partial L}{\partial \phi_{\mu_1 \cdots \mu_a}} \delta\phi_{\mu_1 \cdots \mu_a} + \frac{\partial}{\partial x^\mu}(L\delta x^\mu) \quad . \tag{2.5.13}$$

The right-hand side of this equation has been defined as δL, so that

$$\boxed{\delta L = [\delta L] - \bar{\delta} L} \quad . \tag{2.5.14}$$

This equation, together with the equations (2.5.9) and (2.5.10) for requirements (A) and (B), is the key to the relationship between these requirements and condition (2.5.1).

Clearly, requirements (A) and (B) are sufficient but not necessary conditions that δL be a divergence. Many authors [e.g. Hill(1951), Dothan (1971), Schroder (1968)] derive condition (2.5.1) as a requirement on the transformation that it leave J invariant, but they do so by assuming requirement (A) holds as a property of the Lagrangian, so that from equation (2.5.14),

$$\delta L = - \bar{\delta} L \quad . \tag{2.5.15}$$

That is, under assumption (A), condition (2.5.1) is identical to requirement (B).

Rosen (1971) has a detailed discussion of the points made here, and Dass (1966b) has a derivation of condition (2.5.1).

The advantages of this alternative approach to checking that δJ is zero are that it is both necessary and sufficient, and that it is easy to check any Lagrangian L for this property, given a transformation $(\delta x, \delta\phi)$. The function G^μ [if it exists] can be computed directly from equations (2.5.1) and (2.5.2):

$$\boxed{\sum \frac{\partial L}{\partial \phi_{\mu_1 \cdots \mu_a}} \delta\phi_{\mu_1 \cdots \mu_a} + \frac{\partial}{\partial x^\mu} (L \delta x^\mu) = - \; d_\mu G^\mu \; ,} \tag{2.5.16}$$

and the corresponding conservation law can be found by substitution into conservation equation (2.3.6):

$$\boxed{d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu + G^\mu] = 0 \quad .} \tag{2.5.17}$$

If the left-hand side of equation (2.5.16) will not form a four-divergence, δJ is not zero and Noether's theorem does not apply.

(2.6) GENERALISED NOETHER'S THEOREM

Rosen (1972, 1974a, 1974b) has generalised Noether's theorem so that a Lagrangian is no longer needed to be able to associate an infinitesimal transformation with a conservation law. Since some equations of motion do not possess Lagrangians, this is quite a useful generalisation. All that is required is the equation of motion, and the transformation $\bar{\delta}\phi$. Note that δx is no longer explicitly required, as will be discussed in section (2.7).

Generalised Noether's theorem states that *if the infinitesimal transformation* $\bar{\delta}\phi$ *is such that*

$$\boxed{-\bar{\delta}\phi\, F = d_\mu J^\mu - K} \tag{2.6.1}$$

holds for general field variables ϕ*, where* $K(x,\phi,\phi_\mu \cdots)$ *is zero for solutions and is linearly independent of* F*, and where*

$$F = 0$$

is the equation of motion for the system, then $\bar{\delta}\phi$ *is associated with the conservation equation,*

$$d_\mu J^\mu = 0 \tag{2.6.2}$$

for solutions.

It will be shown that if a Lagrangian exists for a system, generalised Noether's theorem is *equivalent* to the alternative approach to Noether's theorem in section (2.5), *if* K *is identically zero.*

Compare equation (2.6.1) with Noether's relation, equation (2.3.4):

$$-\bar{\delta}\phi E_\phi(L) = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu + G^\mu] \quad . \tag{2.6.3}$$

The only difference is that K is identically zero in Noether's relation [that is, K is not there].

Any transformation $(\delta x, \bar{\delta}\phi)$ associated through Noether's theorem with a conservation law, must be such that Noether's relation holds if the system has a Lagrangian, and hence must be such that generalised Noether's relation (2.6.1) holds with K identically zero.

To see what generalised Noether's theorem implies for a system with a Lagrangian, in particular to see what δL and δJ will look like, recall the definition (2.5.2) of δL:

$$\delta L \equiv \sum_a \frac{\partial L}{\partial \phi_{\mu_1 \cdots \mu_a}} \delta\phi_{\mu_1 \cdots \mu_a} + \frac{\partial}{\partial x^\mu}(L\delta x^\mu) \quad . \tag{2.6.4}$$

Recall the work in section (2.1), in which the right-hand side of equation (2.6.4), as the integrand in J, was rearranged until it was obtained in the form

$$\delta L = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu] + \bar{\delta}\phi E_\phi(L) \quad . \tag{2.6.5}$$

If generalised Noether's relation (2.6.1) is satisfied, substitution for $\bar{\delta}\phi E_\phi(L)$ in equation (2.6.5) gives

$$\delta L = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu - J^\mu] + K \quad , \tag{2.6.6}$$

where K is zero for solutions and is linearly independent of the equation of motion.

If K were identically zero in the above equation, condition (2.5.1) of the alternative approach to Noether's theorem would be satisfied, that is,

$$\delta L = - d_\mu G^\mu \tag{2.6.7}$$

where

$$G^\mu = J^\mu - \pi^\mu(L)\bar{\delta}\phi - L\delta x^\mu \quad , \tag{2.6.8}$$

so that the functional variation of the action integral would be zero.

In general, however, K is zero only for solutions, and δL is equal to a divergence term plus K, so that [recalling equation (2.5.4)]:

$$\delta J = \int_v \delta L dx \quad , \tag{2.6.9}$$

$$= \int_v (d_\mu G^\mu + K) dx \quad , \tag{2.6.10}$$

$$= \int_v K dx \quad , \tag{2.6.11}$$

under the usual assumption that G^μ vanishes on the boundary of v. Hence δJ is zero only for solutions to the equation of motion. This is not saying anything new about δJ, since it is always zero for solutions, under the usual assumption that $\phi_{\mu_1 \ldots \mu_a}$ vanishes, for all $a \geq 0$, on the boundary of v. This follows from Hamilton's principle, and also from the form of equation (2.1.19) for δJ:

$$\delta J = \int_v \{d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu] + \bar{\delta}\phi E_\phi(L)\} dx \quad , \tag{2.6.12}$$

$$= \int_v \bar{\delta}\phi E_\phi(L) dx \quad . \tag{2.6.13}$$

So generalised Noether's theorem gives a slight generalisation of Noether's theorem if a Lagrangian exists, such that δJ is no longer required to be zero, but such that a conservation law still arises from the development of δJ in section (2.1), as follows: recall the result of section (2.1) as expressed in equation (2.6.5),

$$\delta L = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu] + \bar{\delta}\phi E_\phi(L) \quad . \tag{2.6.14}$$

Generalised Noether's theorem requires [see equation (2.6.6)]

$$\delta L = - d_\mu G^\mu + K \tag{2.6.15}$$

where K is zero for solutions and is linearly independent of the equation of motion. Equations (2.6.14) and (2.6.15) yield the conservation law

$$d_\mu J^\mu \equiv d_\mu [\pi^\mu (L) \bar{\delta}\phi + L\delta x^\mu + G^\mu] \tag{2.6.16}$$

$$= 0$$

for solutions, *even though* δJ *is not zero.*

However, as Rosen [(1974a), fourth footnote] points out, this slight generalisation of Noether's theorem to include K which is zero only for solutions does not appear to be of much practical use. *In most practical cases,* K *is identically zero and generalised Noether's theorem is equivalent to Noether's theorem whenever a Lagrangian exists for the system.*

If a Lagrangian does exist for a system, Noether's theorem is a more powerful approach to conservation laws than generalised Noether's theorem, since it allows the prediction of conservation laws from properties of the Lagrangian. For example, as in appendix C, if a Lagrangian is not explicitly dependent on the z-coordinate, it immediately follows from Noether's theorem [the alternative approach] that the z-component of linear momentum is conserved. The usefulness of generalised Noether's theorem is confined to non-Lagrangian systems, for example, those with integro-differential equations of motion.

The property of K that it be linearly independent of the equation of motion is necessary to prevent the association of $\bar{\delta}\phi$ with J^{μ} from being ambiguous. As is mentioned by Rosen [(1974a), second footnote], if K was allowed to depend linearly on the equation of motion, any arbitrary infinitesimal transformation could be trivially associated with a given conservation equation. This is apparent from generalised Noether's relation (2.6.1):

$$-\bar{\delta}\phi E_{\phi}(L) = d_{\mu}J^{\mu} - K \quad . \tag{2.6.17}$$

If a variation $\bar{\delta}\phi$ exists such that generalised Noether's relation holds, take an arbitrary variation $\bar{\delta}_2\phi$. Then from equation (2.6.17),

$$-\bar{\delta}_2\phi E_{\phi}(L) = d_{\mu}J^{\mu} - K + (\bar{\delta}\phi - \delta_2\phi)E_{\phi}(L) \quad , \tag{2.6.18}$$

and if K is allowed to depend linearly on $E_{\phi}(L)$, this equation can be put into the form of generalised Noether's relation:

$$-\bar{\delta}_2\phi E_{\phi}(L) = d_{\mu}J^{\mu} - K_2 \tag{2.6.19}$$

where

$$K_2 = K - (\bar{\delta}\phi - \bar{\delta}_2\phi)E_{\phi}(L) \quad . \tag{2.6.20}$$

In such a way, any arbitrary infinitesimal transformation can be associated with J^{μ} unless K is restricted to be linearly independent of the equation of motion.

Candotti, Palmieri and Vitale (1972) in their treatment of Noether's theorem have as the condition on δL,

$$\delta L = - d_\mu G^\mu + f[E_\phi(L)] \quad , \tag{2.6.21}$$

where f may be linearly dependent on the equation of motion. Comparing this condition to generalised Noether's condition (2.6.6), it is apparent that f equals K. Not surprisingly, these authors conclude that any infinitesimal transformation can be associated through Noether's theorem with any existing conservation law. Their conclusion is a consequence of the ambiguity inherent in their condition (2.6.21).

It should be noted that in any use of conventional or generalised Noether's theorem, the requirements (A) and (B) in section (2.4), condition (2.5.1) or condition (2.6.1) must be shown to hold for general field variables ϕ. Any unwitting use of the equation of motion to help prove that these requirements are met, will succeed in associating any transformation with any existing conservation law. For example, condition (2.6.1) would hold trivially, since both sides are always zero for solutions if J^μ is a conserved vector, enabling any choice of variation $\bar{\delta}\phi$.

(2.7) THE NOETHER FAMILY OF TRANSFORMATIONS

The concept of a Noether family of transformations $(\delta x, \delta\phi)$ helps in comparing the different approaches to

Noether's theorem, and in understanding why δx does not explicitly appear in generalised Noether's theorem.

The transformation associated with a conservation law in generalised Noether's theorem is characterised by $\bar{\delta}\phi$. Since [equation (2.1.11)]

$$\bar{\delta}\phi = \delta\phi - \delta x^{\mu}\phi_{\mu} \quad , \tag{2.7.1}$$

there is a *set of values of* $(\delta x, \delta\phi)$ *with the same value of* $\bar{\delta}\phi$*, and hence associated with the same conserved vector* J^{μ} *and the same* K*, through generalised Noether's theorem.* Candotti, Palmieri and Vitale (1970) refer to this set $\{(\delta x, \delta\phi)\}$ as a *Noether family.*

Rosen (1972) shows by a formal construction that *every Noether family contains a transformation* $(\delta x, \delta\phi)$ *such that the Lagrangian [if it exists] transforms as a scalar density*, which is condition (A) of conventional Noether's theorem, that is,

$$[\delta L] = 0 \tag{2.7.2}$$

as in equation (2.5.9). A proof of this follows.

Recall equation (2.5.14),

$$\delta L = [\delta L] - \bar{\delta} L \quad , \tag{2.7.3}$$

and equation (2.6.6) for δL when generalised Noether's relation (2.6.1) holds,

$$\delta L = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu - J^\mu] + K \quad . \tag{2.7.4}$$

Combine equations (2.7.3) and (2.7.4) to get

$$[\delta L] = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu - J^\mu] + K + \bar{\delta}L \quad . \tag{2.7.5}$$

Without loss of generality, divide $\bar{\delta}L$ into a divergence and a nondivergence term:

$$\bar{\delta}L \equiv d_\mu(\delta L_1^\mu) + \delta L_2 \quad . \tag{2.7.6}$$

Substitute this into equation (2.7.5) to get

$$[\delta L] = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta x^\mu - J^\mu + \delta L_1^\mu] + K + \delta L_2 \quad . \tag{2.7.7}$$

Let a Noether family be associated with the conservation law,

$$d_\mu J^\mu = 0 \tag{2.7.8}$$

for solutions. Take one member $(\delta_1 x, \delta_1\phi)$ of this family, and formally construct another member $(\delta_2 x, \delta_2\phi)$ in terms of the first member, as in Rosen (1972):

$$\delta_2 x^\mu = -\frac{1}{L}[\pi^\mu(L)\bar{\delta}\phi + \delta_1 L_1^\mu - J^\mu] \quad ,$$

$$\delta_2\phi = \bar{\delta}\phi + \delta_2 x^\mu \phi_\mu \quad , \tag{2.7.9}$$

$$\delta_2 L_1^\mu = \delta_1 L_1^\mu \quad ,$$

$$\delta_2 L_2 = - K \quad .$$

Note that $\delta_2 x^\mu$ is dependent on ϕ in this construction. This does not invalidate the result, even though δx has been assumed to be dependent only on x in this chapter.

If the values in equations (2.7.9) are substituted into equation (2.7.7) for $[\delta_2 L]$,

$$[\delta_2 L] = d_\mu[\pi^\mu(L)\bar{\delta}\phi + L\delta_2 x^\mu - J^\mu + \delta_2 L_1^\mu]$$

$$+ K + \delta_2 L_2 \quad , \tag{2.7.10}$$

everything cancels to give

$$[\delta_2 L] = 0 \quad , \tag{2.7.11}$$

and the proof is complete.

If, as is usually the case [see appendix F], K is identically zero,

$$\bar{\delta}_2 L = d_\mu(\delta_2 L_1^\mu) \quad , \tag{2.7.12}$$

that is, condition (B) of conventional Noether's theorem is also met. Hence, *if K is identically zero, requirements (A) and (B) of conventional Noether's theorem will be met by one member* $(\delta_2 x, \delta_2 \phi)$ *of the Noether family.*

Clearly, if K is identically zero, condition (2.5.1) of the alternative approach to Noether's theorem is met by all members of the Noether family. This is proved at equation (2.6.7).

Figure (2.4) summarises in schematic form these relationships between the different approaches to Noether's theorem, in terms of the Noether family.

One feature of Noether's theorem has been brought into the open by generalised Noether's theorem and the concept of a Noether family of transformations. This is the irrelevance of the coordinate transformation δx to the existence of a conservation law, except as it affects $\bar{\delta}\phi$. Changing δx while keeping $\bar{\delta}\phi$ the same will not change the associated conservation law [if it exists], although it will change the values of $[\delta L]$ and $\bar{\delta}L$, as seen earlier in this section.

The irrelevance of δx corresponds to the equivalence of the *active* and *passive* views of a coordinate transformation [Saletan and Cromer (1971)]. For example, a translation can be viewed *passively* as a shifting of the coordinate axes, while the field variables remain stationary. The variation would then be

$$\delta x^{\mu} = \varepsilon^{\mu} \text{ [infinitesimal]}, \ \delta\phi = 0 \tag{2.7.13}$$

and hence

$$\bar{\delta}\phi = - \varepsilon^{\mu}\phi_{\mu}. \tag{2.7.14}$$

FIGURE (2.4)

Schematic of Relationships Between Different Approaches to Noether's Theorem and the Noether Family of Transformations

CONVENTIONAL NOETHER'S THEOREM

Conditions (A) & (B) both satisfied, i.e.

$[\delta L] = 0$,

$\bar{\delta} L = d_\mu G^\mu$.

ALTERNATIVE APPROACH TO NOETHER'S THEOREM

Condition

$\delta L = d_\mu G^\mu$

satisfied.

↑ satisfied iff $K \equiv 0$, for one member $(\delta x, \delta\phi)$ (to Conventional Noether's Theorem)

↑ satisfied iff $K \equiv 0$, for all members $\{(\delta x, \delta\phi)\}$ (to Alternative Approach)

THE NOETHER FAMILY

The set of transformations $\{(\delta x, \delta\phi)\}$ such that

$-\delta\phi E_\phi(L) = d_\mu J^\mu - K$,

and hence associated with:

$d_\mu J^\mu = 0$

for solutions.

↓ satisfied for all allowed K, and for all members $\{(\delta x, \delta\phi)\}$

GENERALISED NOETHER'S THEOREM

Condition $-\delta\phi E_\phi(L) = d_\mu J^\mu - K$ satisfied.

The *active* view of the same translation would be to regard the coordinate axes as stationary, while the field variables are shifted in the opposite direction to that taken by the axes in the passive view. The variation would be

$$\delta x^{\mu} = 0 \quad , \qquad\qquad \delta\phi = - \varepsilon^{\mu}\phi_{\mu} \tag{2.7.15}$$

and hence

$$\bar{\delta}\phi = - \varepsilon^{\mu}\phi_{\mu} \ , \tag{2.7.16}$$

the same as in the passive view.

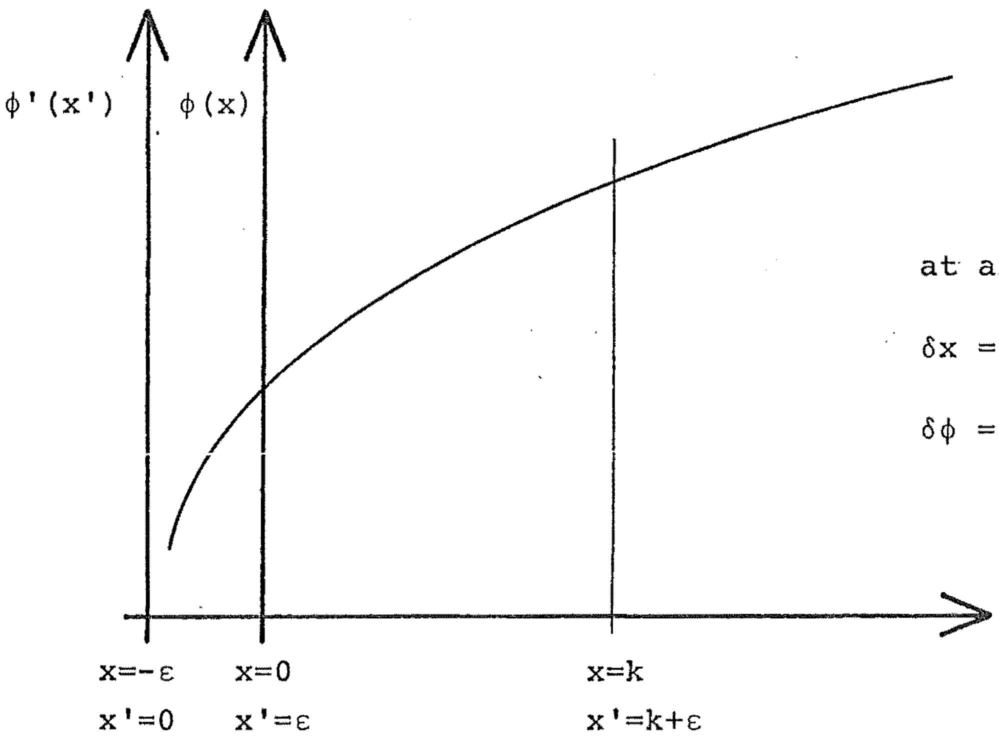
Figure (2.5) gives a pictorial representation of the active and passive views of a translation along the x-axis. No matter which view is taken, the effect of the transformation on the system is the same. Hence it is not surprising that the same conservation law arises from either view. The Noether family consists of the active and passive views of a transformation, and all combinations of these views with the same value of $\bar{\delta}\phi$. That is, the Noether family can be thought of as the set of all the different views of the same transformation.

The Noether family can be used as a means of identification of the conserved density associated with it. As is discussed in appendix (F), the association of a conservation law with a Noether family is unique for the large class of systems with K identically zero. For these systems, the Noether

FIGURE (2.5)

The Active and Passive Views of an x-Translation.

The Passive View of $x' = x + \varepsilon$

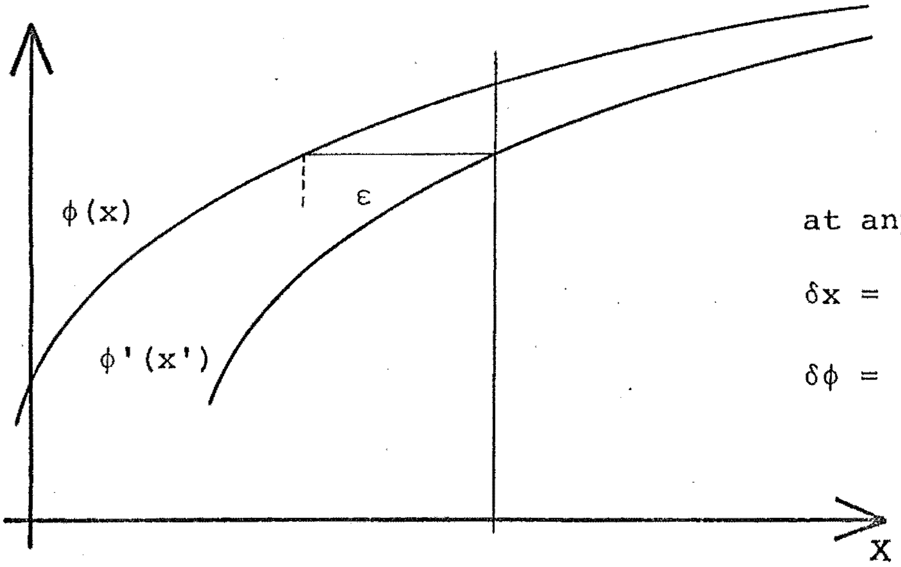


at any point,

$\delta x = \varepsilon$,

$\delta\phi = 0$.

The Active View of $x' = x+\varepsilon$



at any point,

$\delta x = 0$,

$\delta\phi = -\varepsilon\phi_x$.

families are a useful labelling system for their associated conservation laws, and can be used to identify a conservation law as follows:

For example, if a conservation law was found to be associated via Noether's theorem with the variation

$$\bar{\delta}\phi = - \quad \varepsilon\phi_t \tag{2.7.17}$$

where ε is an infinitesimal parameter, the law would be identified as expressing conservation of energy. This follows from the work in appendices B and C, which show that conservation of energy is associated with such a variation.

Note that in the passive view of transformation (2.7.17),

$$\delta t = \varepsilon,\ \delta x^{\mu} = 0 \quad \text{for} \quad x^{\mu} \neq t, \quad \delta\phi = 0 \quad . \tag{2.7.18}$$

This is, the variation can be regarded as a time-translation, hence the association in physics of energy conservation with time-translation invariance.

CHAPTER III

THE KORTEWEG-DE VRIES EQUATION

(3.1) BACKGROUND

The Korteweg-de Vries [KdV] equation has been the subject of much research interest in the last 18 years. It was originally derived by Korteweg and de Vries (1895) for the evolution of long water waves down a rectangular canal, in the form

$$\eta_t = \frac{3}{2}\sqrt{\frac{g}{\ell}}\,\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\eta_{xx}\right) \tag{3.1.1}$$

in one space dimension and time, where η is the surface elevation above the equilibrium level ℓ, α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant and

$$\sigma = \frac{\ell^3}{3} - \frac{T\ell}{\rho g} ,$$

with surface tension T and density ρ.

Despite the general suitability of the KdV equation for describing the unidirectional propagation of small-but-finite amplitude waves in a nonlinear dispersive medium, no new application was discovered until 1960, in the study of collision-free hydromagnetic waves by Gardner & Morikawa (1960). Since then many other applications of the KdV equation have been found, including:

(i) ion-acoustic waves in plasma, by Washimi and Taniuti (1966), and Tappert (1972).

(ii) the anharmonic lattice, a one-dimensional lattice of equal masses coupled by nonlinear springs, also known as the Fermi-Pasta-Ulam problem (1955), by Zabusky (1967, 1973).

(iii) longitudinal dispersive waves in elastic rods, by Nariboli and Sedov (1970).

(iv) pressure waves in a liquid-gas bubble mixture, by Wijngaarden (1968).

(v) The axial component of velocity in a rotating fluid flow down a tube, by Leibovich (1970).

(vi) thermally excited phonon packets in low-temperature nonlinear crystals, by Tappert and Varma (1970).

A large class of nonlinear Galilean-invariant systems, under the assumptions of weak nonlinearity and long wavelength, have been shown to reduce to the KdV equation in the work of Su and Gardner (1969) and of Leibovich and Seebass (1974). A general class of nonlinear systems has been reduced by a perturbation method to the KdV equation by Taniuti and Wei (1968), and Zielke (1974) has shown that a wide class of nonlinear partial differential equations derivable from a variational principle also reduces to the KdV equation.

However, it has been the properties of the KdV *equation and its solutions that have primarily motivated much of the recent research on it.* One of the earliest features to be noticed was the existence of solitary wave solutions which emerge from a collision with the *same* shapes and velocities as before it [Zabusky and Kruskal (1965)]. An *exact method* for solving the KdV equation by means of a linear scattering problem was published by Gardner, Greene, Kruskal and Miura (1967). The existence of an *infinite number of conserved densities*, each of the form of a polynomial in the field variable η and its x-derivatives, was proved in a publication by Miura, Gardner and Kruskal (1968). Wahlquist and Estabrook (1973, 1975) discovered the *Bäcklund transformation* for the KdV equation, which maps solutions to solutions.

These properties of the KdV equation and its solutions are all inter-related. A brief resumé of them follows in sections (3.2) to (3.5). The remainder of this chapter investigates what light can be shed on the infinite number of polynomial conserved densities by the techniques of Noether's theorem. This includes work done by Steudel (1975b), associating an infinitesimal extended Bäcklund transformation with the infinite set of densities, and original work identifying each density as an energy or a momentum density.

(3.2) THE INVERSE SPECTRAL METHOD OF SOLUTION

One of the most remarkable and far-reaching results of research into the KdV equation has been the discovery by Gardner, Greene, Kruskal and Miura (1967) of the *inverse* *spectral method* [I.S.M.] for solving exactly the initial-value problem. That is, given a solution u of the KdV equation at a time t_o, find the solution after a time t_1 has elapsed. This problem was previously only approximately solvable for the KdV equation by numerical methods [e.g., Zabusky and Kruskal (1965)], and the only exact solutions known were the solitary and cnoidal waves [Korteweg and de Vries (1895)]. The I.S.M. has subsequently been applied to other nonlinear equations with some success, and to a broad class of nonlinear equations by Ablowitz, Kaup, Newell and Segur (1973).

The most convenient form of the KdV equation for applying the I.S.M. is

$$\boxed{u_t - 6uu_x + u_{xxx} = 0 \quad .} \tag{3.2.1}$$

The scale transformation

$$u = -\frac{\eta}{2} - \frac{\alpha}{3} \quad ,$$

$$t' = -\frac{1}{2}\left(\frac{g}{\ell\sigma}\right)^{\frac{1}{2}} t \quad ,$$

$$x' = \sigma^{-\frac{1}{2}} x \quad ,$$

transforms equation (3.1.1) to equation (3.2.1), within a constant factor, where the primes have been suppressed in equation (3.2.1). The first step in applying the I.S.M.

to the KdV equation is to regard u(x,t) as the potential in the time-independent one-dimensional Schrodinger scattering equation,

$$\boxed{\psi_{xx}(x,t) - [u(x,t) - \lambda(t)]\psi(x,t) = 0 \quad .} \tag{3.2.2}$$

The time coordinate in equation (3.2.1) appears only parametrically in equation (3.2.2). The *direct scattering problem* in quantum mechanics is to be given the potential u(x), and to find the asymptotic behaviour of $\psi(x)$ at $x = \pm\ \infty$, and the bound state energy levels and normalisation constants, otherwise known as the *scattering data*. The techniques for solving this problem are well-known [see, for example Schiff (1968)].

The *inverse scattering problem* is to find the potential u(x) given the scattering data. The technique will be outlined here in the usual quantum mechanical context, that is, with t fixed.

The key to the use of the I.S.M. to solve the KdV equation is that *if the potential* u(x,t) *is allowed to evolve in the parameter* t *according to the* KdV *equation, the discrete eigenvalues* λ_m *do not change,* and the other scattering data change in a simple way with the parameter t.

The Schrodinger equation (3.2.2) has a set of continuous eigenvalues

$$\lambda = k^2 \quad , \tag{3.2.3}$$

and discrete eigenvalues

$$\lambda_m = -k_m^2, \quad m = 1, \ldots N. \tag{3.2.4}$$

The eigenfunctions of the discrete eigenvalues are normalised:

$$\psi_m = c_m(t)\exp(-k_m x), \text{ for } x \to \infty \quad , \tag{3.2.5}$$

and for the continuous eigenvalues, the scattering problem is set as

$$\begin{aligned} \psi &\approx \exp(-ikx) + b\exp(ikx), \text{ for } x \to \infty , \\ \psi &\approx a\exp(-ikx), \quad \text{for } x \to -\infty , \end{aligned} \tag{3.2.6}$$

and it can be shown that

$$|a|^2 + |b|^2 = 1 \quad .$$

The scattering data are then the quantities k_m, c_m and $b(k)$.

The key discovery by Gardner, Greene, Kruskal and Miura (1967) was that the scattering data depend in a very simple way on the parameter t if $u(x,t)$ is required to be a solution of the KdV equation (3.2.1). Their work showed that

$$\begin{aligned} \lambda_m(t) &= \lambda_m(0) , \\ a(k,t) &= a(k,0) , \\ b(k,t) &= b(k,0)\exp(8ik^3 t) , \\ c_n(t) &= c_n(0)\exp(4k_n^3 t) , \end{aligned} \tag{3.2.7}$$

where $\lambda_m(0)$, $b(k,0)$ and $c_n(0)$ are determined from the initial data for the KdV equation, $u(x,0)$, by the *direct scattering problem*. It is assumed that k is independent of t, to find a and b.

The *inverse scattering problem* can then be solved for $\lambda(t)$, $b(k,t)$ and $c_n(t)$ to find [exactly] $u(x,t)$ at any time t. This problem has been studied by Gel'fand and Levitan (1955), Kay and Moses (1956), and Levinson (1953). They show that the solution is

$$u = -2\frac{d}{dx}K(x,x) \quad , \tag{3.2.8}$$

where K satisfies the Gel'fand-Levitan integral equation

$$K(x,y) + B(x+y) + \int_x^\infty B(y+z)K(x,z)dz = 0 \quad . \tag{3.2.9}$$

The kernel B is given in terms of the scattering data:

$$B(x) = \sum_{m=1}^{N} c_m^2 \exp(-k_m x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp(ikx)dk \quad . \tag{3.2.10}$$

Note that dependence on the parameter t in B and K has been suppressed in these equations.

To summarise, the I.S.M. for solving the KdV equation is as follows: beginning with an initial solution to the KdV equation, $u(x,0)$, solve the direct scattering problem (3.2.2) to find the scattering data $\lambda_m(0)$, $b(k,0)$ and $c_n(0)$. Allow the scattering data to evolve according to equations (3.2.7) for parametric time t. Substitute the new scattering data $\lambda_m(t)$, $b(k,t)$ and $c_n(t)$ into equation (3.2.10) to find $B(x;t)$, and solve the [linear] Gel'fand-Levitan integral equation (3.2.9) to find $K(x,x)$ and hence $u(x,t)$.

The I.S.M. for solving the KdV equation is shown diagrammatically in figure (3.1).

Although explicit solutions $u(x,t)$ cannot be obtained in general, exact solutions can be given for the cases when $b(k,0)$ is zero [no reflection of ψ from the potential u], as in Gardner, Greene, Kruskal and Miura (1974). The main simplification achieved is that all equations to be solved are linear.

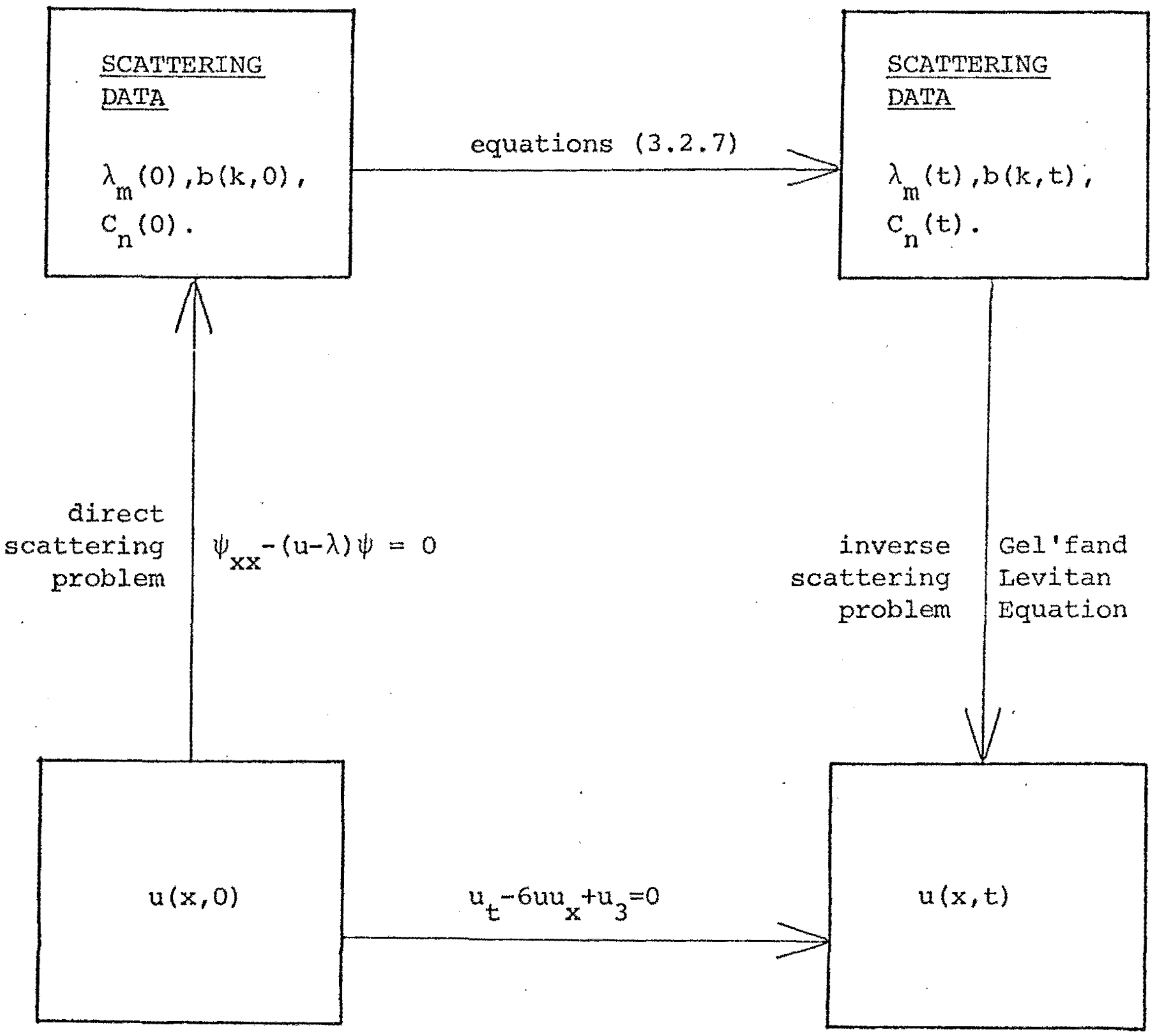
The form of B in equation (3.2.10) suggests a link with the *Fourier transform technique* for solving linear equations. Ablowitz, Kaup, Newell and Segur (1974) show that the I.S.M. does reduce to the Fourier technique on linearisation of the equations of motion, and that *the I.S.M. may be regarded as a generalisation to nonlinear systems of the Fourier transform method.*

(3.3) SOLITON SOLUTIONS

A *soliton* is a particular type of solitary wave or localised travelling wave. The working definition of Scott, Chu and McLaughlin (1973) will be adopted, that *a soliton is a solitary wave solution of a wave equation which asymptotically preserves its shape and velocity upon collision with other solitary waves.*

FIGURE (3.1)

Diagram of the Inverse Spectral Method for Solving the KdV Equation



Such solutions are well-known in linear wave systems without dispersion, since all waves interact linearly and continue on unaffected in shape and velocity after the interaction. It was Zabusky and Kruskal (1965) who discovered in a computer study of the KdV equation that solitary wave solutions emerge from a collision with the same shape and velocity as before the collision. This was an unexpected result as the collision is a non-linear interaction, and this motivated much of the subsequent research into the KdV equation. Also unexpected was the discovery in the same paper that a large class of initial conditions will break up into a number of solitons, the largest in front, with an oscillatory tail spreading to the rear.

The inverse spectral method has been used *a posteriori* to predict accurately both of these features of solitary wave solutions of the KdV equation [Gardner, Greene, Kruskal and Miura, (1967)], and to express in closed form, solutions describing any finite number of solitons in interaction. Soliton solutions correspond to the zero reflection case $b(k,0) = 0$ in the I.S.M., which as mentioned in section (2.2) is exactly solvable.

An interesting connection between solitons and conservation laws was noticed by Zabusky and Kruskal (1965). They found that a soliton solution solved the variational problem that the variation δI_3 be zero subject to the integral I_2 being constant, where I_3 and I_2 are the integrals of the first two conserved densities of the KdV equation. The variation δI_3 is a *functional variation*, which by the analysis in section (2.2) is zero for appropriate boundary conditions if and only if the Euler-Lagrange operator, operating on the integrand in I_3, is zero. So the variation of

$$I_3 \equiv \int \left(\frac{u^3}{3} - u_x^2\right) dx \tag{3.3.1}$$

is zero if and only if

$$E_u\left(\frac{u^3}{3} - u_x^2\right) = u^2 + 2u_{xx} = 0 \quad . \tag{3.3.2}$$

The variation of

$$I_2 \equiv \int \frac{u^2}{2}\, dx \tag{3.3.3}$$

is similarly zero if and only if

$$u = 0 \quad . \tag{3.3.4}$$

Using Lagrange multiplier λ, set δI_3 to zero subject to fixed I_2, so that

$$\delta I_3 + \lambda\, \delta I_2 = 0 \quad . \tag{3.3.5}$$

This holds if and only if

$$u^2 + 2u_{xx} + \lambda u = 0 \quad . \tag{3.3.6}$$

This is a second-order ordinary differential equation in u, having as a solution the soliton solution to the KdV equation:

$$u = \frac{3\lambda}{2} \operatorname{sech}^2 \left[\sqrt{\frac{\lambda}{8}} \left(x - \frac{\lambda}{2} t\right) + \theta \right] \quad , \tag{3.3.7}$$

where θ is some arbitrary phase. The amplitude of this soliton is $\frac{3\lambda}{2}$, and its velocity is $\frac{\lambda}{2}$.

Lax (1976) has proved what Kruskal (1974) conjectured, that *if* I_n *is extremised* [that is, δI_n set to zero] *subject to constraints on the lower integrals, a* 2(n-2)th-*order ordinary differential equation is obtained*, with (n-2) parameters [Lagrange multipliers] representing the positions of (n-2) wave solutions. *One solution for this equation is that representing* (n-2) *solitons in interaction.*

(3.4) BACKLUND TRANSFORMATIONS

Bäcklund transformations have been found for the KdV equation [and other nonlinear equations of interest] which *transform solutions to solutions* [Wahlquist and Estabrook (1973 & 1975)]. The KdV equation is written in the form

$$w_t - 6w_x^2 + w_{xxx} = 0, \quad 2w_x \equiv u \, , \tag{3.4.1}$$

for which the Bäcklund transformations are [Steudel (1975)]:

$$\boxed{\begin{aligned} \bar{w}_x + w_x &= (\bar{w} - w)^2 - k^2 \quad , \\ \bar{w}_t + w_t &= -2(\bar{w} - w)(\bar{w}_{xx} - w_{xx}) + (\bar{w}_x - w_x)^2 \\ &\quad + 3[(\bar{w} - w)^2 - k^2]^2 \quad . \end{aligned}} \tag{3.4.2}$$

Although equations (3.4.2) can be put in first-order form, solution for w, given $\bar{w}$, is very difficult in general. However, using the *theorem of permutability* due to Bianchi [see Eisenhart (1960)], N-soliton solutions can be generated from the vacuum solution $\bar{w} = 0$, as is shown by Wahlquist and Estabrook (1973).

Another use of the Bäcklund transformation is to provide an alternative way of deriving the scattering equation for solving the KdV equation with the I.S.M., with possibilities for use as a general technique for finding an I.S.M. for solving nonlinear equations [Miura (1976)].

Bäcklund transformations are also intimately related to the existence of infinitely many conservation laws. In particular, Steudel (1975b) has shown that *the infinite set of conservation laws of the* KdV *equation arise from invariance of the* KdV *equation under so-called infinitesimal extended Bäcklund transformations*, via Noether's theorem. This will be investigated in more detail in subsection (3.6).

(3.5) THE GENERALISED KDV EQUATIONS

When it was discovered that the discrete eigenvalues of the Schrodinger equation (3.2.2) are invariant if the potential $u(x,t)$ is allowed to evolve according to the KdV equation, a search was made for other evolution equations with this same property [*generalised* KdV equations]. Lax (1968) discovered a generalised process for obtaining these equations in the form

$$\boxed{u_t = K_n(u) \ , \quad n = 0, 1, \ldots} \tag{3.5.1}$$

where $K_n(u)$ is a nonlinear differential [in x] operator, using inverse spectral theory.

Gardner (1970) found them in the form

$$\boxed{u_t = -d_x \left(\frac{\delta I_n}{\delta u}\right) \ , \quad n = 2, 3, \ldots} \tag{3.5.2}$$

where the term in brackets is the variational derivative of I_n with respect to u, and I_n is the integral over x of the nth polynomial conserved density of the KdV equation.

Lenard [see Gardner, Greene, Kruskal and Miura, (1974)] found an alternative form:

$$\boxed{u_t = - d_x A_n \ , \quad n = 1, 2, \ldots} \tag{3.5.3}$$

where the A_n are polynomial conserved densities of the KdV equation generated by the recursion relation

$$\boxed{(d_x^2 + \frac{2}{3} u + \frac{1}{3} u_x \int) d_x A_n = d_x A_{n+1} \quad ,} \tag{3.5.4}$$

with

$$A_1 = u \ ,$$

where the KdV equation is written in the form

$$u_t + uu_x + u_{xxx} = 0 \quad . \tag{3.5.5}$$

The first few *generalised* KdV equations are

$$u_t = - u_x \quad , \tag{3.5.6}$$

$$u_t = - uu_x - u_3 \quad , \qquad \text{[the KdV equation]} \ ,$$

$$u_t = - \frac{5}{6} u^2 u_1 - \frac{5}{3} uu_3 - \frac{10}{3} u_1 u_2 - u_5 \quad ,$$

where

$$u_1 \equiv u_x, \ u_2 \equiv u_{xx} \ , \quad \text{etc.}$$

These equations share the same infinite set of conserved densities, that is, the set originally discovered for the KdV equation. This property was proved by Miura, Gardner and Kruskal (1967) even before the *generalised* KdV equations were found: they showed that any equation sharing the property of the KdV equation that the discrete eigenvalues of the Schrodinger equation are invariant if the potential u(x,t) is a solution, will also share the same set of conserved densities.

Clearly, they will share the additional properties of being solvable by the Schrodinger scattering problem, of having soliton solutions, and of having Bäcklund transformations taking solutions to solutions. However, they have not been found to be of any physical interest in themselves, except for the case of the KdV equation.

(3.6) AN INFINITE SET OF CONSERVATION LAWS

The first five conserved densities and fluxes for the KdV equation in the form

$$u_t + uu_1 + u_3 = 0 \tag{3.6.1}$$

are presented in appendix G. That there existed an infinite number of conserved quantities for a nonlinear system was a matter of some interest to researchers, since it is a property usually associated with linear systems [see appendix H]. Noether's theorem gives some insight into conservation laws by associating them with an infinitesimal transformation on the system, and can often *identify* a conserved density by the nature of its associated transformation. For example, an infinitesimal time translation identifies an energy density, an infinitesimal rotation identifies an angular momentum density, and an infinitesimal phase or gauge transformation identifies a charge density. The work of Steudel (1975b) *has identified the infinite set of conservation laws for the* KdV *equation via Noether's theorem, as associated with infinitesimal extended Bäcklund transformations*. The following subsection looks at his work.

(3.6.i) The Infinitesimal Extended Bäcklund Transformation and Conservation Laws

Steudel takes the first half of transformation (3.4.2) [we are once again working with the KdV equation in the form (3.4.1)]:

$$w'_x + w_x = (w'-w)^2 - k^2 \tag{3.6.2}$$

representing a transformation of solution w to solution w' of the KdV equation. Rearranging this to solve for (w'-w), and specifying positive square root,

$$w'-w = k[1-k^{-2}(w'_x + w_x)]^{\frac{1}{2}} \tag{3.6.3}$$

which defines the *extended Bäcklund transformation:*

$$\boxed{w' = B_k w \quad .} \tag{3.6.4}$$

As Steudel points out, and as is mentioned at the end of section (2.6), *this transformation must be defined for a general class of functions* w, not just for solutions to the KdV equation, in order that Noether's theorem can be used. In this sense it is an *extension* of the usual Bäcklund transformation (3.4.2), which is defined for solutions only. Steudel writes w' as a Laurent series

$$w' = k + w + \sum_{i=1}^{\infty} A_i k^{-i} \quad , \tag{3.6.5}$$

where the coefficients A_i may be determined by recurrence after substituting into equation (3.6.4):

$$A_1 = w_x \quad , \tag{3.6.6}$$

$$A_i = \frac{1}{2} d_x A_{i-1} - \frac{1}{2} \sum_{r=1}^{i-2} A_r A_{i-r-1} \quad .$$

In the following, Steudel uses the abbreviations

$$s(ij) \equiv w(i) + w(j), \quad d(ij) \equiv w(i) - w(j), \tag{3.6.7}$$

where

$$\begin{aligned} w(1) &= B_k w(0) \quad , \\ w(2) &= B_{k+\varepsilon}\, w(0) \quad , \\ w(3) &= B_{k+\varepsilon}\, w(1) = B_k w(2) \quad , \\ w(4) &= B_k w(1) \, , \end{aligned} \tag{3.6.8}$$

represented diagrammatically in figure (3.2). The equivalence of $B_{k+\varepsilon}\, B_k\, w(0)$ to $B_k\, B_{k+\varepsilon}\, w(0)$ is a result of the theorem of permutability due to Bianchi (see Eisenhart, 1960).

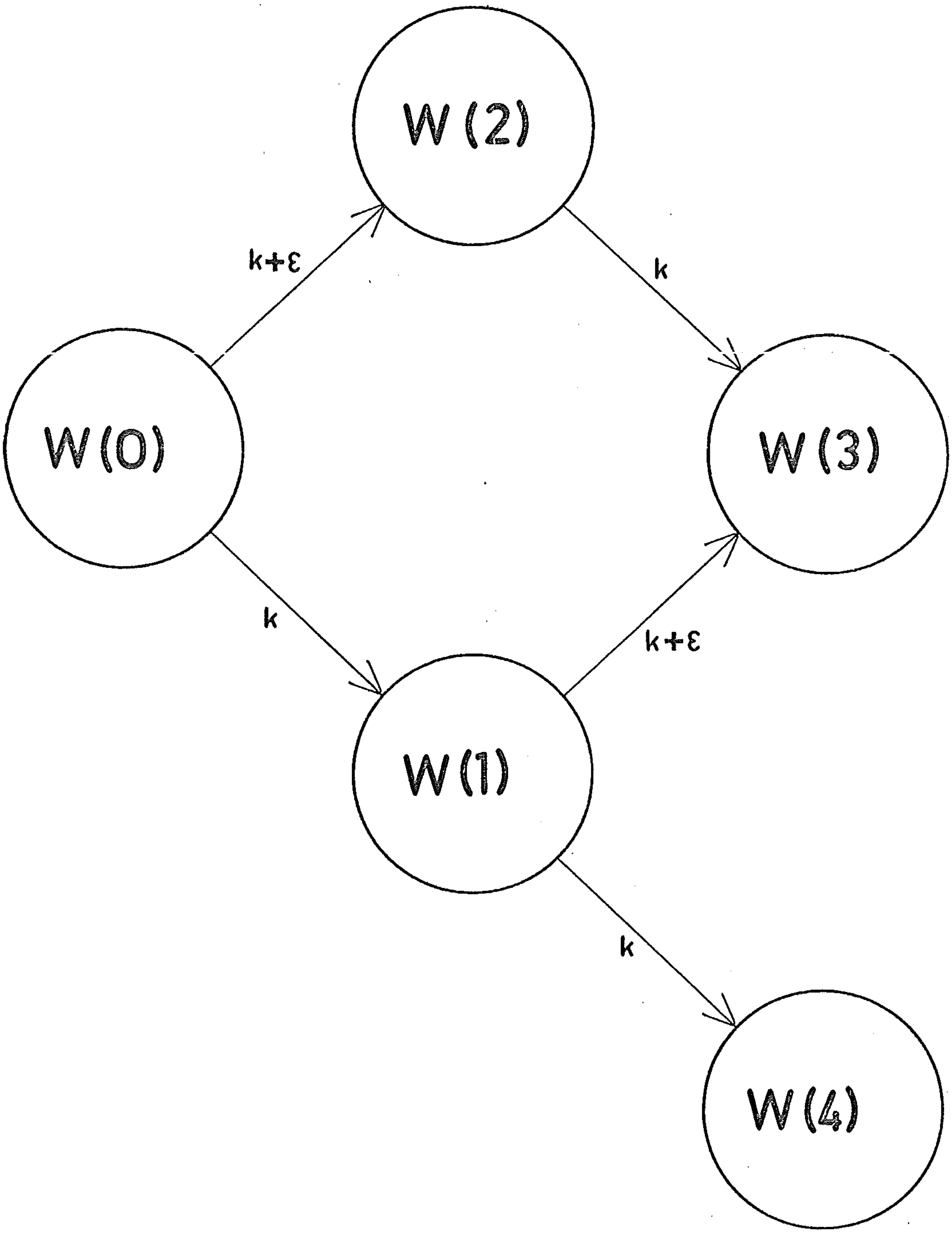
The variation of the Lagrangian

$$L = \frac{1}{2} w_1 w_t - \frac{1}{2} w_2^2 - 2w_1^3 \tag{3.6.9}$$

under the finite transformation

FIGURE (3.2)

The Extended Bäcklund Transformations



$$\bar{\delta} w = B_k w(0) - w(0) \quad , \tag{3.6.10}$$

is

$$L[w(1)] - L[w(0)] = T^{10}_t + X^{10}_x \quad , \tag{3.6.11}$$

where

$$T^{10} = -d(10)\left(\frac{d^2(10)}{12} + \frac{k^2}{4}\right) \quad ,$$

$$X^{10} = d(10)\left[-\frac{4d^4(10)}{5} - 2k^2 + w_x(d^2(10) - k^2) - 2w_x^2 + \frac{1}{4}s_t(10)\right] \quad . \tag{3.6.12}$$

To use Noether's theorem, the variation of the Lagrangian under the infinitesimal extended Bäcklund transformation

$$\delta w = B_{k+\varepsilon} B_{-k} w(1) - w(1) \quad , \tag{3.6.13}$$

$$= d(21) \quad ,$$

where $\varepsilon << k$, can be found by varying the vector (T^{10}, X^{10}) with respect to k $[\delta k = \varepsilon]$ where w(0) is assumed fixed, since

$$\delta w = B_{k+\varepsilon} w(0) - B_k w(0) \quad ,$$

$$= [B_{k+\varepsilon} w(0) - w(0)] - [B_k w(0) - w(0)] \quad . \tag{3.6.14}$$

This is an infinitesimal variation in k of the first finite transformation (3.6.10). Then it is found that

$$\delta L = L[w(2)] - L[w(1)] \quad ,$$

$$= \varepsilon(T_t + X_x) \quad , \tag{3.6.15}$$

where, after removing trivially conserved parts of the vectors,

$$\varepsilon T = -\frac{k\varepsilon}{2} d(40) - \frac{1}{2} w_x d(21),$$

$$\varepsilon X = 2d(21)[-w_x^2 - k^2 w_x - k^4 + d(10)d(14)(k^2 - w_x)$$

$$+ \frac{1}{8} s_t(10)] \quad . \tag{3.6.16}$$

This gives the variation of the Lagrangian density as a divergence, as in equation (2.5.16). Hence a conservation law results. Noting that

$$\pi^t(L)\bar{\delta}w = \frac{\partial L}{\partial w_t} \bar{\delta}w = \frac{1}{2} w_x \bar{\delta}w \quad , \tag{3.6.17}$$

$$\pi^x(L)\bar{\delta}w = \frac{\partial L}{\partial w_x} \bar{\delta}w + \frac{\partial L}{\partial w_{xx}} \bar{\delta}w_x - \left(\frac{\partial L}{\partial w_{xx}}\right)_x \bar{\delta}w \quad ,$$

$$= (\frac{1}{2} w_t - 6w_x^2 + w_3)\bar{\delta}w - w_{xx} \bar{\delta}w_x \quad , \tag{3.6.18}$$

the conservation law, equation (2.5.17), becomes

$$\boxed{[d(40)]_t + [4d(40)(k^2 - w_x)]_x = 0 \quad .} \tag{3.6.19}$$

Recall that

$$d(40) = B_k w(1) - B_{-k}\, w(1) \quad , \tag{3.6.20}$$

$$= 2k + \sum_n 2A_{2n+1}\, k^{-(2n+1)} , \tag{3.6.21}$$

using the Laurent expansion (3.6.5). Since k is an arbitrary parameter, the coefficient of each inverse power of k is a conserved density, yielding an infinite number of conserved densities. Note that only odd powers of k^{-1} contribute, so that the A_{2n+1} generated by formula (3.6.6) are conserved densities for the KdV equation, associated with the infinitesimal extended Bäcklund transformation d(21). These are the same conserved densities [within trivially conserved densities] as those discovered by Miura, Gardner and Kruskal (1967).

Steudel's work can be seen to associate the infinite set of conservation laws of the KdV equation with the infinitesimal extended Bäcklund transformations that have been found for the KdV equations. It was the search for a simpler identification of these conservation laws that motivated the work in the following sections of this chapter.

(3.7) CONSERVED DENSITIES OF THE KDV EQUATION IDENTIFIED AS ENERGY DENSITIES

It will be shown that the conserved densities of the KdV *equation can be identified via Noether's theorem as* *energy densities of higher-order equations, whose conservation laws must be obeyed by solutions to the* KdV *equation.* This provides a simpler identification than that of Steudel, one with a physical interpretation as an energy density of some enveloping higher-order system. The higher-order system envelopes the system described by the KdV equation in the sense that it contains the solution set for the KdV system as part of its own solution set. As will be shown in later chapters, this technique can also be applied to other nonlinear equations of interest.

The higher-order equations are obtained from the KdV equation in the form

$$\boxed{\phi_{1t} + \phi_1\phi_2 + \phi_4 = 0 \quad ,} \tag{3.7.1}$$

where

$$\phi_1 \equiv \phi_x \equiv \frac{\partial\phi(x,t)}{\partial x} \quad , \quad \phi_{1t} \equiv \frac{\partial^2\phi}{\partial t\partial x} \quad ,$$

$$\phi_2 \equiv \phi_{xx} \quad , \qquad \text{etc.,}$$

by operating on it n times in succession with

$$\boxed{H \equiv d_x^2 + \frac{2}{3}\,\phi_1 + \frac{1}{3}\,\phi_2 \int^x dx \quad ,} \tag{3.7.2}$$

where the lower limit of the integral is chosen so that ϕ and its derivatives vanish on that boundary. The higher-order KdV equations are then

$$\boxed{H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = 0 \quad , \quad n = 1,2, \ldots} \quad . \tag{3.7.3}$$

These are integro-differential equations, of no known physical interest in themselves, except that solutions to the KdV equation are also solutions to equations (3.7.3). Hence conservation laws for equations (3.7.3) must be obeyed by solutions to the KdV equation.

It will be shown that equations (3.7.3) have conserved energy densities which are identical to the conserved densities of the KdV equation. It is well known that energy conservation is associated with invariance under time translation [see, for example, H. Rund (1966), H. Sagan (1969) and appendix B]. Recall generalised Noether's theorem, section (2.6) that if

$$- \bar{\delta}\phi \; F = d_\mu J^\mu - K \quad , \tag{3.7.4}$$

where

$$K[\phi] = 0$$

for solutions to

$$F = 0 \quad , \tag{3.7.5}$$

and K is linearly independent of F, then the infinitesimal transformation $\bar{\delta}\phi$ is associated with the conservation law

$$d_\mu J^\mu = 0 \quad . \tag{3.7.6}$$

It will be proved that, for any field variable ϕ, and for n = 0,1, ...

$$\boxed{\phi_t H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = d_t(-A_{n+3}) + d_x X_{n+3} \quad ,} \tag{3.7.7}$$

where the A_i are the conserved densities for the KdV equation. Since the field variation for time translation is [see appendix B]:

$$\bar{\delta}\phi = -\varepsilon\phi_t \quad , \tag{3.7.8}$$

where ε is some infinitesimal parameter, generalised Noether's theorem then identifies A_{n+3} as a conserved energy density [within a minus sign] for the equation of motion (3.7.3). Furthermore, the $K[\phi]$ referred to in equation (3.7.4) is identically zero in equation (3.7.7), so that the energy vector $(-A_{n+3}, X_{n+3})$ is a conserved vector of the first kind, as in appendix D.

Equation (3.7.7) will be proved in two parts. In section (3.7.i), the *integro-differential* part will be proved, that is, it will be shown that

$$\boxed{\phi_t H^n(\phi_{1t}) = d_x F_n, \; n = 0,1, \ldots} \tag{3.7.9}$$

where F_n is *equivalent* to an acceptable function of ϕ and its derivatives and integrals of these. *Equivalent* is used here in the context of appendix D, meaning *equal within a trivially conserved vector*. *Acceptable* is used in the sense that *a flux is acceptable if it vanishes with ϕ and its derivatives*. This is explained in more detail later in this section, at equations (3.7.29) to (3.7.34).

In section (3.7.ii) the remaining *partial differential* part of equation (3.7.7) will be proved, that is, for n=0,1, ...

$$\boxed{\phi_t H^n(\phi_1\phi_2 + \phi_4) = d_t(-A_{n+3}) + d_x(\bar{X}_{n+3}) ,} \tag{3.7.10}$$

where $\bar{X}_{n+3}$ is a function of ϕ and its derivatives. It was proved that $H^n(\phi_1\phi_2 + \phi_4)$ is a polynomial in ϕ_1 and its x derivatives by Gardner, Greene, Kruskal and Miura (1974), hence the label *partial differential part.*

It is convenient to separate the problem into two parts because section (3.7.ii) can be simply proved, using the results of recent research into the KdV equation, and section (3.7.i) is a separate problem by reason of its integro-differential nature and/or the presence of a time derivative.

(3.7.i) The Integro-Differential Part

Equation (3.7.9) will be proved by induction. Assuming that

$$\phi_t H^i(\phi_{1t}) = d_x F_i \quad \text{for} \quad i = 0,1, \ldots , n , \tag{3.7.11}$$

it will be proved that

$$\phi_t H^{n+1}(\phi_{1t}) = d_x F_{n+1} \quad . \tag{3.7.12}$$

In the following, the operand ϕ_{1t} will be suppressed for simplicity of notation: it is understood that H^r means $H^r(\phi_{1t})$, and $(H^i H^k)$ means H^{k+i}.

The left-hand side of equation (3.7.12) is

$$\phi_t H^{n+1} = \phi_t (d_x^2 + \frac{2}{3}\phi_1 + \frac{1}{3}\phi_2 \int dx) H^n \,, \tag{3.7.13}$$

$$= d_x[\phi_t d_x H^n - \phi_{1t} H^n + \frac{1}{3}(\int \phi_2 \phi_t) \int H^n]$$

$$+ (H^n) \int H \quad , \tag{3.7.14}$$

noting that

$$\int H = \int^x H(\phi_{1t})\,dx = \phi_{2t} + \frac{2}{3}\phi_1\phi_t - \frac{1}{3}\int^x \phi_2\phi_t\,dx \;. \tag{3.7.15}$$

The process which has been started in equation (3.7.14) can be continued with the use of the following lemma:

LEMMA: that

$$\boxed{H^{k+1} \int H^i = d_x Q_{i,k} + H^k \int H^{i+1}}$$

for all

$$i \leqslant n, \quad k < n \quad . \tag{3.7.16}$$

PROOF:

$$H^{k+1} \int H^i = [(d_x^2 + \frac{2}{3}\phi_1 + \frac{1}{3}\phi_2 \int dx) H^k] \int H^i \quad . \tag{3.7.17}$$

Rearranging the right-hand side in much the same way as was done in equation (3.7.13), it becomes

$$d_x[(d_x H^k) \int H^i - (H^i)(H^k) + \frac{1}{3}(\int H^k)(\int \phi_2 \int H^i)]$$

$$+ [(d_x^2 + \frac{2}{3}\phi_1 - \frac{1}{3}\int \phi_2) \int H^i] H^k \quad . \tag{3.7.18}$$

Noting that integration by parts gives

$$\int (HH^i) = (d_x^2 + \frac{2}{3}\phi_1 - \frac{1}{3}\int \phi_2)\int H^i \quad , \tag{3.7.19}$$

equation (3.7.17) becomes

$$H^{k+1}\int H^i = d_x\, Q_{i,k} + H^k \int H^{i+1} \quad , \tag{3.7.20}$$

where

$$Q_{i,k} \equiv (d_x H^k)\int H^i - (H^i)(H^k) + \frac{1}{3}(\int \phi_2 \int H^i)(\int H^k). \tag{3.7.21}$$

This completes the proof of the lemma.

The right-hand side of equation (3.7.14) can be written as

$$d_x\, Q_{o,n} + (H^n)\int H \quad . \tag{3.7.22}$$

Applying the lemma to this expression, equation (3.7.14) becomes

$$\phi_t H^{n+1} = d_x Q_{o,n} + d_x Q_{1,(n-1)} + (H^{n-1})\int H^2 \quad . \tag{3.7.23}$$

Repeatedly applying the lemma to the last term will lead to the result

$$\phi_t\, H^{n+1} = \sum_{i=0}^{n} d_x\, Q_{i,(n-i)} + \phi_{1t}\int H^{n+1} \quad , \tag{3.7.24}$$

which is

$$\phi_t H^{n+1} = d_x\left[\sum_{i=0}^{n} Q_{i,(n-i)} + \phi_t \int H^{n+1}\right] - \phi_t H^{n+1} \quad . \tag{3.7.25}$$

This implies that

$$\phi_t H^{n+1} = d_x \frac{1}{2}\left[\sum_{i=0}^{n} Q_{i,(n-i)} + \phi_t \int H^{n+1}\right] \quad . \tag{3.7.26}$$

That is, equation (3.7.12) is proved. Since

$$\phi_t \; H^0(\phi_{1t}) = \phi_t \phi_{1t} = d_x(\tfrac{1}{2}\; \phi_{1t}^2) \quad , \tag{3.7.27}$$

the induction process is started, and equation (3.7.9) is proved for $n = 0,1, \ldots$

It will be noted that F_n contains terms which are *pure integral terms*, for example

$$[\int H^k(\phi_{1t})]\,[\int \phi_2 \int H^i(\phi_{1t})] \quad , \tag{3.7.28}$$

where the integrals are taken from $-\infty$ to x. *Such terms are in general quite unacceptable as flux terms in a conservation equation, since they give rise to source or sink terms as follows:*

To obtain a conserved density from a conservation equation

$$T_t + X_x = 0 \quad , \tag{3.7.29}$$

recall that it is integrated over x from $-\infty$ to $+\infty$, so that

$$d_t \int_{-\infty}^{\infty} T dx + [X]_{-\infty}^{\infty} = 0 \quad , \tag{3.7.30}$$

and under the assumption that ϕ and its derivatives vanish at $x = \pm\,\infty$, a flux X containing *acceptable* terms [that is, terms which vanish with ϕ and its derivatives] will itself vanish on the boundary, giving

$$d_t \int_{-\infty}^{\infty} T\, dx = 0 \tag{3.7.31}$$

If, however, X contains a term

$$\int_{-\infty}^{x} F(\phi) dx \tag{3.7.32}$$

then

$$\Big[X\Big]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} F(\phi) dx \quad , \tag{3.7.33}$$

which is not zero in general under the previous assumptions, so that equation (3.7.30) becomes

$$d_t \int_{-\infty}^{\infty} T dx = - \int_{-\infty}^{\infty} F(\phi) dx \quad . \tag{3.7.34}$$

The term (3.7.32) has given rise to a source/sink term for the density T, and T is not in general conserved.

One integral which appears in all such integral terms in F_n is

$$\boxed{\int_{-\infty}^{x} H^{i}(\phi_{1t})dx \quad .} \tag{3.7.35}$$

It will be shown that this term is equivalent [in the sense of appendix D] to a polynomial in ϕ_1 and its x-derivatives. This will complete the proof of section (3.7.i), because such a polynomial will vanish on the boundary of integration by assumption, and since all integral terms contain such a term, they will also vanish on the boundary.

To prove that term (3.7.35) is equivalent to a polynomial, a result obtained by Gardner, Greene, Kruskal and Miura (1974) will be used. They derived a recursion formula for the conserved densities A_i of the KdV equation which uses the same operator H as is used in this thesis to generate higher-order KdV equations:

$$Hd_x A_n = d_x A_{n+1} \tag{3.7.36}$$

where

$$A_1 = \phi_x \quad . \tag{3.7.37}$$

This can be used to obtain

$$H^2 d_x A_2 = Hd_x A_3 = d_x A_4 \quad , \tag{3.7.38}$$

and in general, noting that

$$d_x A_2 = \phi_1\phi_2 + \phi_4 \quad , \tag{3.7.39}$$

this becomes

$$H^n(\phi_1\phi_2 + \phi_4) = d_x A_{n+2} \quad . \tag{3.7.40}$$

Using this result,

$$H^i(\phi_{1t}) = H^i(\phi_{1t} + \phi_1\phi_2 + \phi_4) - H^i(\phi_1\phi_2 + \phi_4) \, , \tag{3.7.41}$$

$$= H^i(\phi_{1t} + \phi_1\phi_2 + \phi_4) - d_x A_{i+2} \quad . \tag{3.7.42}$$

Hence

$$\int_{-\infty}^{x} H^i(\phi_{1t})dx = -A_{i+2} + \int_{-\infty}^{x} H^i(\phi_{1t} + \phi_1\phi_2 + \phi_4)dx \quad . \tag{3.7.43}$$

The last term is zero for solutions to

$$H^r(\phi_{1t} + \phi_1\phi_2 + \phi_4) = 0 \quad , \quad r = 0,1, \ldots , i \, , \tag{3.7.44}$$

thus it is a vector term of the *fourth kind* [see appendix D] when the equation of motion is any of equations (3.7.44). Such a vector term is *trivial*, and the term (3.7.35) is equivalent to $-A_{i+2}$, a polynomial in ϕ_x and its x-derivatives. This completes the proof of section (3.7.i).

(3.7.ii) The Partial Differential Part

To prove equation (3.7.10), the extensive literature on the KdV equation will be drawn upon to yield a Lagrangian density with an Euler-Lagrange equation of the form

$$H^n(\phi_1\phi_2 + \phi_4) = 0 \quad . \tag{3.7.45}$$

Noether's theorem will then be used as in appendix C to find an energy density for equation (3.7.45), and to show that this energy density is such that equation (3.7.10) follows.

The research that has been done on the *generalised* KdV equations can be used to obtain quite simply a Lagrangian density for equation (3.7.45). In section (3.5), two different forms for the same set of *generalised* KdV equations [within a constant factor] are presented. One is that due to Gardner (1970),

$$u_t = -d_x G_{n+1} \quad , \quad n = 1, 2, \ldots \tag{3.7.46}$$

where G_n is the gradient or variational derivative of the integral

$$\int A_n(u)\,dx \quad . \tag{3.7.47}$$

The gradient can be written in terms of the integrand as

$$G_n(u) = E_u[A_n(u)] \quad . \tag{3.7.48}$$

The form of these equations due to Lenard [see Gardner, Greene, Kruskal and Miura (1974)] is

$$u_t = -d_x A_n \quad , \qquad n = 1,2, \ldots \tag{3.7.49}$$

Hence within a constant factor

$$d_x A_n = d_x E_u(A_{n+1}) \quad . \tag{3.7.50}$$

The following lemma will be used to find a Lagrangian density for equation (3.7.45):

LEMMA: that

$$d_x E_u[F(u)] = -E_\phi[F(\phi_x)] \qquad (3.7.51)$$

where F(u) is some [possibly nonlinear] differential function, and $u \equiv \phi_x$.

PROOF: by definition (2.1.1),

$$d_x E_u[F(u)] = d_x\left\{\sum_{a=0} (-1)^a d_{\mu_1} \cdots d_{\mu_a} \frac{\partial F(u)}{\partial u_{\mu_1 \cdots \mu_a}}\right\}, \qquad (3.8.52)$$

$$= d_x\left\{\sum_{a=0} (-1)^a d_{\mu_1} \cdots d_{\mu_a} \frac{\partial F(\phi_x)}{\partial \phi_{x\mu_1 \cdots \mu_a}}\right\}, \qquad (3.7.53)$$

$$= -\sum_{a=0} (-1)^{a+1} d_x d_{\mu_1} \cdots d_{\mu_a} \frac{\partial F(\phi_x)}{\partial \phi_{x\mu_1 \cdots \mu_a}}, \qquad (3.7.54)$$

$$= -E_\phi[F(\phi_x)] \quad . \qquad (3.7.55)$$

Recall equation (3.7.40):

$$H^n(\phi_1\phi_2 + \phi_4) = d_x A_{n+2} \quad . \qquad (3.7.56)$$

Using equations (3.7.50) and (3.7.51), this becomes

$$\boxed{H^n(\phi_1\phi_2 + \phi_4) = E_\phi[-A_{n+3}(\phi_x)] \quad .} \qquad (3.7.57)$$

That is, $-A_{n+3}$ is a Lagrangian density for equation (3.7.45). The results in appendix C show that if the Lagrangian density has no explicit time-dependence, the variation of the Lagrangian is zero under $\bar{\delta}\phi = -\varepsilon\phi_t$, and the energy density is conserved and is given by

$$T^{tt} = \pi^t(L)\phi_t - L \quad , \tag{3.7.58}$$

$$= A_{n+3}(\phi_x) \quad . \tag{3.7.59}$$

Hence Noether's relation, equation (2.0.4), gives

$$\phi_t H^n(\phi_1\phi_2 + \phi_4) = d_t(-A_{n+3}) + d_x[\pi^x(A_{n+3})\phi_t] , \tag{3.7.60}$$

and section (3.7.ii) is completed.

Combining the results (3.7.12) and (3.7.60) gives the equation

$$\boxed{\phi_t H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = d_t(-A_{n+3}] + d_x[F_n + \pi^x(A_{n+3})\phi_t] \quad ,} \tag{3.7.61}$$

that is, *the conserved energy density of the* nth *higher-order* KdV *equation is* A_{n+3}, *one of the polynomial conserved densities for the* KdV *equation.* Note that since the set of polynomial conserved densities for the KdV equation is unique [Kruskal, Miura and Gardner (1970)], it was not really necessary to obtain the explicit form of the conserved energy density for the nth higher-order equation, to be able to claim its equivalence to the (n+3)rd. polynomial conserved density of the KdV equation. All that was needed was to show this energy density is a polynomial in ϕ and its x derivatives.

(3.8) COROLLARY FOR GENERALISED KDV EQUATIONS

The result in section (3.7.ii) leads to the corollary that *each of the infinite number of polynomial conserved densities for the generalised* KdV *equations is an energy density for one of the generalised* KdV *equations.* This may be stated in terms of generalised Noether's theorem:
If

$$\phi_{xt} - K_n(\phi_x) = 0 \quad , \qquad n = 0,1, \ldots \tag{3.8.1}$$

are the *generalised* KdV equations in Lax's form [section (3.5)], then

$$\boxed{\phi_t[\phi_{xt} - K_n(\phi_x)] = d_t(-A_{n+2}) + d_x Y_{n+2} \quad .} \tag{3.8.2}$$

Relation (3.8.2) identifies $(-A_{n+2})$ as an energy density of the nth *generalised* KdV equation.

To prove equation (3.8.2), Lenard's form for these equations in section (3.5) will be used:

$$\phi_{xt} + d_x A_{n+1} = 0 \ , \qquad n = 0,1, \ldots \tag{3.8.3}$$

The recursion relation (3.7.40) can be used, giving

$$\phi_{xt} + H^{n-1}(\phi_1\phi_2 + \phi_4) = 0, \quad n = 0,1 \ldots \tag{3.8.4}$$

Hence

$$\phi_t[\phi_{xt} + H^{n-1}(\phi_1\phi_2 + \phi_4)] = d_x(\tfrac{1}{2}\phi_t^2) + \phi_t H^{n-1}(\phi_1\phi_2 + \phi_4), \tag{3.8.5}$$

and using result (3.7.60), this becomes

$$\phi_t\,[\phi_{xt} + H^{n-1}(\phi_1\phi_2 + \phi_4)] = d_t(-A_{n+2}) + d_x[\tfrac{1}{2}\phi_t^2 + \pi^x(A_{n+2})\phi_t]\ . \tag{3.8.6}$$

Since Lenard's form (3.8.3) is equal to Lax's form (3.8.1), equation (3.8.2) is proved within a constant factor, with

$$Y_n = \pi^x(A_n)\phi_t + \frac{1}{2}\phi_t^2 \quad . \tag{3.8.7}$$

(3.9) CONSERVED DENSITIES OF THE KDV EQUATION IDENTIFIED AS MOMENTUM DENSITIES

It will be shown by a procedure similar to that in section (3.7) that *the conserved densities of the* KdV *equation can be given an alternative identification as momentum densities of the higher-order* KdV *equations,* via Noether's theorem. That is, it will be shown that

$$\boxed{\phi_x H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = d_t(A_{n+2}) + d_x(Z_{n+2})} \tag{3.9.1}$$

for n = 0,1, ...

Noether's theorem will then identify A_{n+2} as a momentum density for the equation

$$H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = 0 \quad , \tag{3.9.2}$$

associated with the transformation $\bar{\delta}\phi = -\varepsilon\phi_x$.

As in section (3.7), the proof of equation (3.9.1) will be undertaken in two sections. The first will deal with the integro-differential part, showing that

$$\boxed{\phi_x H^n(\phi_{1t}) = d_t(A_{n+2}) + d_x(\bar{Z}_{n+2}) \quad .} \tag{3.9.3}$$

The second will deal with the partial differential part, showing that

$$\boxed{\phi_x H^n(\phi_1\phi_2 + \phi_4) = d_x(\bar{Z}_{n+1}) \quad .} \tag{3.9.4}$$

(3.9.i) <u>The Integro-Differential Part</u>

The method for proving equation (3.9.3) is similar to that adopted in section (3.7.i). The left-hand side is

$$\phi_x H^n(\phi_{1t}) = \phi_x(d_x^2 + \tfrac{2}{3}\phi_1 + \tfrac{1}{3}\phi_2 \int dx)H^{n-1} \quad , \tag{3.9.5}$$

$$= d_x[\phi_x d_x H^{n-1} - \phi_2 H^{n-1} + \tfrac{1}{6}\phi_1^2 \int H^{n-1}]$$

$$+ H^{n-1}(\phi_{1t}) \int H(\phi_2) \quad , \tag{3.9.6}$$

suppressing the ϕ_{1t} operand in $H^{n-1}(\phi_{1t})$, and noting that

$$\int H(\phi_2) = \phi_3 + \frac{1}{2}\phi_1^2 \quad . \tag{3.9.7}$$

The following lemma will be useful in proving equation (3.9.3):

LEMMA: that

$$\boxed{H^{k+1}(\phi_{1t}) \int H^j(\phi_2) = d_x R_{k,j} + H^k(\phi_{1t}) \int H^{j+1}(\phi_2) \quad .} \tag{3.9.8}$$

PROOF: once again suppressing the ϕ_{1t} operand in

$$H^k(\phi_{1t}) \quad ,$$

$$(H^{k+1}) \int H^j(\phi_2) = (d_x^2 + \frac{2}{3}\phi_1 + \frac{1}{3}\phi_2 \int) H^k \int H^j(\phi_2) \quad . \tag{3.9.9}$$

The right-hand side is equal to

$$d_x[(d_x H^k) \int H^j(\phi_2) - (H^k)H^j(\phi_2) + \frac{1}{3}(\int H^k) \int \phi_2 \int H^j(\phi_2)]$$
$$+[d_x H^j(\phi_2) + \frac{2}{3}\phi_1 \int H^j(\phi_2) - \frac{1}{3}\int \phi_2 \int H^j(\phi_2)]H^k \quad , \tag{3.9.10}$$

which gives

$$(H^{k+1}) \int H^j(\phi_2) = d_x R_{k,j} + (H^k) \int H^{j+1}(\phi_2) \quad , \tag{3.9.11}$$

where

$$R_{k,j} \equiv (d_x H^k) \int H^j(\phi_2) - (H^k) H^j(\phi_2) + \frac{1}{3}(\int H^k) \int \phi_2 \int H^j(\phi_2) \, . \tag{3.9.12}$$

Applying the lemma to the last term in equation (3.9.6) gives

$$\phi_x H^n(\phi_{1t}) = d_x(R_{(n-1),0} + R_{(n-2),1}) + H^{n-2}(\phi_{1t}) \int H^2(\phi_2) \ . \tag{3.9.13}$$

This process may be repeated to obtain

$$\phi_x H^n(\phi_{1t}) = \sum_{i=0}^{n-1} d_x \ R_{(n-i-1),i} + \phi_{1t} \int H^n(\phi_2) \tag{3.9.14}$$

$$= d_x\left[\sum_{i=0}^{n-1} R_{(n-i-1),i} + \phi_t \int H^n(\phi_2)\right] - \phi_t H^n(\phi_2) \ . \tag{3.9.15}$$

Noting that

$$H^n(\phi_2) = H^{n-1}(\phi_1\phi_2 + \phi_4) \quad , \tag{3.9.16}$$

and using equation (3.7.60), equation (3.9.15) becomes

$$\phi_x H^n(\phi_{1t}) = d_x\left[\sum_{i=0}^{n-1} R_{(n-i-1),i} + \phi_t \int H^n(\phi_2) - \pi^x(A_{n+2})\phi_t\right]$$

$$+ d_t[\ A_{n+2}] \quad , \tag{3.9.17}$$

and equation (3.9.3) is proved, with

$$\bar{Z}_k \equiv \sum_{i=0}^{k-2} R_{(k-i-2),i} + \phi_t \int H^{k-1}(\phi_2) - \pi^x(A_{k+1})\phi_t \quad . \tag{3.9.18}$$

The integral terms in $R_{k,j}$ have been shown to be equivalent to acceptable flux terms in section (3.7.i).

(3.9.ii) The Partial Differential Part

Equation (3.9.4) can be proved by using the result (3.7.57), that

$$H^{n}(\phi_1\phi_2 + \phi_4) = E_{\phi}[-A_{n+3}(\phi_x)] \quad . \tag{3.9.19}$$

Since there is no explicit x-dependence in A_i, the results in appendix C give the variation in this Lagrangian density as zero under the x-translation $\bar{\delta}\phi = -\varepsilon\phi_x$, so that Noether's result, equation (2.0.4), gives

$$\phi_x H^{n}(\phi_1\phi_2 + \phi_4) = d_t\{\pi^t[A_{n+3}(\phi_x)]\phi_x\}$$

$$+ d_x\{\pi^x(A_{n+3})\phi_x - A_{n+3}\} \quad , \tag{3.9.20}$$

$$= d_x[\pi^x(A_{n+3})\phi_x - A_{n+3}] \quad . \tag{3.9.21}$$

Hence equation (3.9.4) is proved, with

$$\bar{Z}_i = \pi^x(A_{i+2})\phi_x - A_{i+2} \quad . \tag{3.9.22}$$

This completes the proof of equation (3.9.1), with

$$Z_n = \sum_{i=0}^{n-2} R_{(n-i-2),i} + \phi_t \int H^{n-1}(\phi_2) - \pi^x(A_{n+1})\phi_t$$

$$+ \pi^x(A_{n+2})\phi_x - A_{n+2} \quad . \tag{3.9.23}$$

(3.10) APPLICATION TO THE LINEARISED KDV EQUATION

The *linearised* KdV equation is

$$\boxed{\phi_{1t} + \phi_4 = 0 \quad .} \tag{3.10.1}$$

Multiplication by ϕ_t gives

$$\phi_t(\phi_{1t} + \phi_4) = d_t(\tfrac{1}{2}\phi_2^2) + d_x(\tfrac{1}{2}\phi_t^2 + \phi_3\phi_t - \phi_2\phi_{1t}) \ , \tag{3.10.2}$$

which gives the energy conservation law

$$d_t T^{tt} + d_x T^{tx} = 0 \quad , \tag{3.10.3}$$

with energy density

$$T^{tt} = \frac{1}{2}\phi_2^2 \quad . \tag{3.10.4}$$

In the same manner as in appendix H, there is an infinity of conservation laws derivable from equation (3.10.3):

$$d_t T^{tt}[\phi_n] + d_t T^{tx}[\phi_n] = 0 \ , \quad n = 0,1,2, \ldots \tag{3.10.5}$$

where, for example,

$$T^{tt}[\phi_n] = \frac{1}{2}\phi_{n+2}^2 \quad , \tag{3.10.6}$$

and where

$$\phi_n \equiv d_x^n \phi \quad . \tag{3.10.7}$$

The conservation laws (3.10.5) arise from the equation

$$\phi_{nt}[\phi_{(n+1)t} + \phi_{n+4}] = d_t\, T^{tt}[\phi_n] + d_x T^{tx}[\phi_n] \quad , \qquad (3.10.8)$$

which is equation (3.10.2) with every ϕ replaced by ϕ_n. The left-hand side of equation (3.10.8) is

$$\phi_{nt}[\phi_{(n+1)t} + \phi_{n+4}] = \phi_{nt}\, d_x^n[\phi_{1t} + \phi_4] \ , \qquad (3.10.9)$$

$$= (-1)^n \phi_t (d_x^2)^n [\phi_{1t} + \phi_4]$$

$$+ d_x\left[\sum_{i=1}^{n} (-1)^{i+1}\, \phi_{(n-i)t}\, d_x^{n+i-1}(\phi_{1t} + \phi_4)\right] . \qquad (3.10.10)$$

Equations (3.10.8) and (3.10.10) imply that

$$\boxed{\phi_t(-d_x^2)^n(\phi_{1t} + \phi_4) = d_t T^{tt}[\phi_n] + d_x\{T^{tx}[\phi_n] + Y_n\} \ ,} \qquad (3.10.11)$$

where

$$Y_n \equiv \sum_{i=1}^{n} (-1)^{i+1}\, \phi_{(n-i)t}\, d_x^{n+i-1}(\phi_{1t} + \phi_4) \quad . \qquad (3.10.12)$$

Equation (3.10.11) is simply a linearised version of result (3.7.61),

$$\phi_t H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = d_t \bar{-}(A_{n+3}) + d_x[F_n + \pi^x(A_{n+2})\phi_1] \ . \qquad (3.10.13)$$

Hence the infinite set of conservation laws derivable from equation (3.10.3) for the linearised KdV equation can be identified as energy densities of the higher-order linearised KdV equations:

$$(d_x^2)^n(\phi_{1t} + \phi_4) = 0 , \qquad n = 0,1,2 \ldots , \tag{3.10.14}$$

whose solution sets contain that of the linearised KdV equation.

Hence the technique of section (3.7) [and, incidentally, of section (3.9)] can be regarded as a generalisation to the nonlinear case of what happens in the linear case.

CHAPTER IV

THE MODIFIED KORTEWEG-DE VRIES EQUATION

(4.1) BACKGROUND

Recall the KdV equation in the form of equation (3.6.1),

$$\boxed{u_t + uu_1 + u_3 = 0 \quad .} \tag{4.1.1}$$

The *modified* KdV equation is the equation with the simplest modification of the nonlinear term in the KdV equation:

$$\boxed{u_t + u^2u_1 + u_3 = 0 \quad .} \tag{4.1.2}$$

One application of this equation is in the study of anharmonic lattices [Zabusky (1967)].

There is a transformation from solutions of equation (4.1.2) to solutions of equation (4.1.1), due to Miura (1968). The transformation is such that if v satisfies (4.1.2), then u, defined by

$$\boxed{u \equiv v^2 \pm (-6)^{\frac{1}{2}} v_x \quad ,} \tag{4.1.3}$$

satisfies the KdV equation (4.1.1). By explicit calculation, in fact,

$$u_t + uu_1 + u_3 = (2v \pm (-6)^{\frac{1}{2}} \frac{\partial}{\partial x})(v_t + v^2v_1 + v_3) \, . \tag{4.1.4}$$

As Miura says, "*it is rare and surprising to find a transformation between two simple nonlinear differential equations of independent interest*". The transformation (4.1.3) led to the proof of the existence of an infinite number of polynomial conserved densities for the KdV equation and for the modified KdV equation, by Miura, Gardner and Kruskal (1967), and to the inverse spectral method of solution of the KdV equation by Gardner, Greene, Kruskal and Miura (1967).

The first five conserved densities and fluxes of the modified KdV equation, due to Miura, Gardner and Kruskal (1968), are presented in appendix I.

In this chapter it is attempted to shed some light on these conservation laws by using Noether's theorem. This is done in a similar manner to chapter III on the KdV equation. In section (4.2) the work done by Steudel (1975c) is briefly presented, in which an infinitesimal extended Bäcklund transformation is associated with these conservation laws, using Noether's theorem. In section (4.3), the conservation laws are given a simpler identification as energy densities of higher-order equations, whose solutions contain those of the modified KdV equation, using Noether's theorem and generalised Noether's theorem. An alternative identification as momentum densities is given in section (4.4).

(4.2) BACKLUND TRANSFORMATIONS AND CONSERVATION LAWS

Recall section (3.6.i) in which Steudel's (1975b) work on the KdV equation was presented. Steudel has also applied his technique for associating an infinite set of conservation laws, via Noether's theorem, with an infinitesimal extended Bäcklund transformation, to the modified KdV equation [Steudel (1975c)]. A brief summary of his results follows.

Perform the infinitesimal extended Bäcklund transformation

$$\boxed{\delta w = B_{a+\varepsilon} B_a^{-1}(w) - w \quad ,} \tag{4.2.1}$$

where ε is an infinitesimal parameter, and

$$B_a(w) \equiv \sum_{\nu=0}^{\infty} A_\nu(w) a^\nu \quad , \tag{4.2.2}$$

where the series expansion (4.2.2), valid in the limit that parameter a tends to zero, is such that

$$d_x[B_a(w) + w] = a^{-1} \sin[B_a(w) - w]. \tag{4.2.3}$$

That is, the A_ν are found recursively, by substitution into equation (4.2.3).

The form of the modified KdV equation chosen by Steudel is

$$w_{1t} + 6w_1 w_2 + w_4 = 0 \ , \tag{4.2.4}$$

which can be obtained from equation (4.1.2) by the substitution

$$u \equiv 6w_x \quad . \tag{4.2.5}$$

The Lagrangian for equation (4.2.4) is

$$L = \tfrac{1}{2}\, w_1 w_t - \tfrac{1}{2} w_2^2 + \tfrac{1}{2} w_1^4 \quad . \tag{4.2.6}$$

Steudel finds the variation of the Lagrangian under the infinitesimal transformation (4.2.1),

$$\delta L \equiv \sum_a \frac{\partial L}{\partial w_{\mu_1 \cdots \mu_a}}\, \delta w_{\mu_1 \cdots \mu_a} \quad , \tag{4.2.7}$$

to be in the form of a divergence. Hence Noether's theorem gives an associated conservation law, the density of which turns out to be equivalent to

$$T = 1 - \cos d_{10} \quad , \tag{4.2.8}$$

where

$$d_{10} \equiv w - B_{-a}(w) \quad . \tag{4.2.9}$$

Substitution of the series expansion for B_{-a} into this conserved density gives an infinite number of conserved densities, one for each coefficient of the powers of a.

The first two conserved densities obtained from equation (4.2.8) are

$$T_1 = \tfrac{1}{2}w_1^2 \quad , \tag{4.2.10}$$

$$T_2 = \tfrac{1}{2}w_1^4 + w_1 w_2 + \tfrac{1}{2}w_2^2 \quad .$$

These are found to be linear combinations of the conserved densities found by Miura, Gardner and Kruskal (1968) [see appendix I], after converting back to the variable u, using equation (4.2.5).

It seems likely that the set of densities generated by equation (4.2.8) is linearly related to that found by Miura, Gardner and Kruskal. No rigorous uniqueness proof for the set of conservation laws of the modified KdV equation has yet been published, although these authors do promise such a proof. Uniqueness does follow, however, in a simple fashion, from the uniqueness of the set of polynomial conserved densities found for the KdV equation [Kruskal, Miura and Gardner (1970)] and from the transformation (4.1.3). To see this, define the *rank* r of a density [as in Kruskal, Miura & Gardner (1970)] as the sum of the *degree* m and half the *index* n of any term in that density,

$$r \equiv m + \tfrac{1}{2}n$$

where the degree is the total number of factors in a term, and the index is the total number of differentiations in a term, as defined in Kruskal, Miura & Gardner (1970). The conserved densities of the KdV and modified KdV equation are of uniform rank, and are of integral rank only [Miura, Gardner and Kruskal (1968)].

Assume that two independent conserved densities of the same rank exist for the modified KdV equation.

Transforming each of these with transformation (4.1.3),

$$u = v^2 \pm (-6)^{\frac{1}{2}} v_x \tag{4.2.11}$$

will yield two independent conserved densities of the same rank for the KdV equation. Since this contradicts the uniqueness result of Kruskal, Miura and Gardner (1970), *the set discovered for the modified* KdV *equation must be unique*.

Hence the set of conserved densities generated by Steudel, from equation (4.2.8), is linearly dependent on the set generated by Miura, Gardner and Kruskal (1968).

(4.3) CONSERVED DENSITIES OF THE MODIFIED KdV EQUATION IDENTIFIED AS ENERGY DENSITIES

The technique used in section (3.7) to identify the conserved densities of the KdV equation as energy densities of higher-order enveloping equations, will be applied here to the modified KdV equation. Since there has not been the same extensive research done on the modified KdV equation, this application cannot lean as heavily as in chapter III on results obtained by other researchers. However, the approach will be similar to that in section (3.7).

The modified KdV equation is put in the form

$$\boxed{\phi_{1t} + \phi_1^2\phi_2 + \phi_4 = 0 \quad ,} \tag{4.3.1}$$

which is equation (4.1.2) with the substitution

$$u \equiv \phi_1 \quad . \tag{4.3.2}$$

The higher-order equations are obtained by operating on the modified KdV equation (4.3.1) n times in succession with the operator

$$\boxed{M \equiv d_x^2 + \frac{2}{3}\phi_1^2 + \frac{2}{3}\phi_2 \int^x dx\ \phi_1,} \tag{4.3.3}$$

where the lower limit of the integral is chosen such that ϕ and its derivatives vanish on that boundary.

The higher-order modified KdV equations are

$$\boxed{M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) = 0 \quad , \qquad n = 0,1,\ \ldots} \tag{4.3.4}$$

Clearly their solution sets contain that of the modified KdV *equation, so that their conservation laws govern solutions to the modified* KdV *equation.* Equations (4.3.4) are integro-differential, and are of no known physical interest in themselves.

It will be shown in this section that these higher-order modified KdV equations have conserved energy densities which are polynomials in ϕ and its x-derivatives, and which are linearly independent of each other. The first five such energy densities may be compared with the first five polynomial conserved densities in appendix I, from which it will be apparent that these two sets are equivalent.

This also follows from the uniqueness of the set of polynomial conserved densities for the modified KdV equation, since these energy densities constitute such a set.

As indicated at equation (3.7.7), it is sufficient to show that

$$\boxed{\phi_t \; M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) = d_t T_n + d_x X_n \quad ,} \tag{4.3.5}$$

in order to establish the conservation law

$$d_t T_n + d_x X_n = 0 \quad , \tag{4.3.6}$$

and identify it as expressing conservation of energy for the equation of motion

$$M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) = 0 \quad . \tag{4.3.7}$$

Equation (4.3.5) will be proved in two parts. In the *partial differential* part, it will be shown that

$$\boxed{\phi_t M^n(\phi_1^2\phi_2 + \phi_4) = d_t T_n + d_x X_n' \quad ,} \tag{4.3.8}$$

where T_n is a polynomial in ϕ and its x-derivatives. This will be done by deriving a Lagrangian for the equation

$$M^n(\phi_1^2\phi_2 + \phi_4) = 0 \quad , \tag{4.3.9}$$

and then using Noether's theorem.

In the *integro-differential* part, it will be shown that

$$\boxed{\phi_t M^n(\phi_{1t}) = d_x Q^n \quad ,} \tag{4.3.10}$$

where Q^n is equivalent to a function which is zero whenever ϕ and its derivatives are zero. This will be done using the same method as that adopted in chapter III to prove equation (3.7.9).

A lemma will first be proved for operator M which will be useful in both parts to follow:

LEMMA: that the following relation holds:

$$\boxed{fM^n g_1 = d_x R^n(f,g) - gM^n f_1} \quad , \tag{4.3.11}$$

where R^n is equivalent to a function which vanishes with ϕ and its derivatives for those functions f and g of ϕ and its derivatives which satisfy the requirements that

1. either $\quad M^i f_1 \approx d_x G^i(\phi, \phi_\mu, \ldots)$

   or $\quad M^{n-i} g_1 \approx d_x G^{n-i}(\phi, \phi_\mu \ldots)$ $\quad$ (4.3.12)

for each $i < n$, where $\approx$ means *is equivalent to* in the sense of appendix D, and where G^i is any function which vanishes with its arguments.

2. either $\quad \phi_1 M^i f_1 \approx d_x F^i(\phi, \phi_\mu \ldots)$

   or $\quad \phi_1 M^{n-i-1} \approx d_x F^{n-i-1}(\phi, \phi_\mu \ldots)$ $\quad$ (4.3.13)

for each $i < n$, where F^i is any function which vanishes with its arguments.

PROOF: $$fM^n g_1 = fM(M^{n-1}g_1) \quad , \tag{4.3.14}$$

$$= f[(M^{n-1}g_1)_2 + \tfrac{2}{3}\phi_1^2 M^{n-1}g_1 + \tfrac{2}{3}\phi_2 \int \phi_1 M^{n-1}g_1] \quad , \tag{4.3.15}$$

$$\begin{aligned} &= d_x[f(M^{n-1}g_1)_1 - f_1M^{n-1}g_1 + f_2 \int M^{n-1}g_1 \\ &\quad + \tfrac{2}{3}\phi_1 f \int \phi_1 M^{n-1}g_1 - \tfrac{2}{3}(\int \phi_1 f_1)(\int \phi_1 M^{n-1}g_1) \\ &\quad + \tfrac{2}{3}\phi_1 (\int M^{n-1} g_1)(\int \phi_1 f_1) - (\int M^{n-1}g_1)(\int Mf_1)] \\ &\quad + (\int Mf_1)M^{n-1}g_1 \quad . \end{aligned} \tag{4.3.16}$$

The x-derivative expression in square brackets is equivalent to a function which vanishes with ϕ and its derivatives if:

1. either $\quad Mf_1 \approx d_x G^1(\phi,\phi_\mu \ldots)$,

   or $\quad M^{n-1}g_1 \approx d_x G^{n-1}(\phi,\phi_\mu \ldots)$. (4.3.17)

2. either $\quad \phi_1 f_1 \approx d_x F^0(\phi,\phi_\mu \ldots)$,

   or $\quad \phi_1 M^{n-1}g_1 \approx d_x F^{n-1}(\phi,\phi_\mu \ldots)$. (4.3.18)

Applying this process repeatedly to the last term in equation (4.3.16), that equation becomes

$$fM^n g_1 = d_x R^{n'}(f,g) + (\int M^n f_1)g_1 \quad , \tag{4.3.19}$$

$$= d_x[R^{n'}(f,g) + (\int M^n f_1)g] - gM^n f_1 \quad . \tag{4.3.20}$$

where the x-divergence expression in square brackets is equivalent to a function which is zero whenever ϕ and its derivatives are zero if the conditions (4.3.12) and (4.3.13) hold. This completes the proof of the lemma, with the relabelling

$$R^n(f,g) \equiv R^{n'}(f,g) + \left(\int M^n f_1\right)g \quad . \tag{4.3.21}$$

(4.3.i) THE PARTIAL DIFFERENTIAL PART

The first step in proving equation (4.3.8),

$$\boxed{\phi_t M^n(\phi_1^2\phi_2 + \phi_4) = d_t T_n + d_x X_n' \quad ,} \tag{4.3.22}$$

is to prove the following lemma:

LEMMA: *that the expression*

$$M^i(\phi_1^2\phi_2 + \phi_4) \tag{4.3.23}$$

is a polynomial in ϕ_1 *and its* x-*derivatives, for all positive integers* i.

PROOF: is by induction. Assume that expression (4.3.23) is such a polynomial for all $i < n$. Using the previous lemma (4.3.11),

$$\phi_1 M^{n-1}(\phi_1^2\phi_2 + \phi_4) = d_x R^{n-1} - (\phi_1^3 + \phi_3)M^{n-1}(\phi_2) . \tag{4.3.24}$$

The integrals in R^{n-1} are all of the form

$$\int M^i\phi_2 \text{ or } \int M^i(\phi_1^2\phi_2 + \phi_4) \text{ or } \int \phi_1 M^i\phi_2 \quad \text{or}$$

$$\int \phi_1 M^i(\phi_1^2\phi_2 + \phi_4) \quad , \tag{4.3.25}$$

for $i < n-1$. Noting that

$$M\phi_2 = \phi_1^2\phi_2 + \phi_4 \ , \tag{4.3.26}$$

it can be seen that some of these integrals are equal to others, so that the only integrals to consider in R^{n-1} are

$$\int M^i(\phi_1^2\phi_2 + \phi_4) \quad \text{and} \quad \int \phi_1 M^i(\phi_1^2\phi_2 + \phi_4) \quad , \tag{4.3.27}$$

where $i < n-1$. Since

$$M^i = d_x[M_x^{i-1} + \tfrac{2}{3}\phi_1\int \phi_1 M^{i-1}], \tag{4.3.28}$$

if M^i is a polynomial, so also is

$$\int \phi_1 M^{i-1} \quad ,$$

and hence so also is

$$\int M^i \quad .$$

Hence by this reasoning, the assumption that expression (4.3.23) is a polynomial for $i < n$ means that the integrals (4.3.27) in R^{n-1} are also polynomials, so that R^{n-1} is composed of terms which are polynomials in ϕ_1 and its x-derivatives.

Noting that

$$\int M\phi_2 = \phi_1^3 + \phi_3 \quad , \tag{4.3.29}$$

equation (4.3.24) can be written as

$$\phi_1 M^{n-1}(\phi_1^2\phi_2 + \phi_4) = d_x[R^{n-1} - (\int M\phi_2)(\int M^{n-1}\phi_2)] + (\int M^{n-1}\phi_2)M\phi_2 \ . \tag{4.3.30}$$

The above reasoning also gives the new integral terms in the square brackets of equation (4.3.30) as polynomials. Applying the lemma (4.3.11) to the last term in equation (4.3.30), that equation becomes

$$\phi_1 M^{n-1}(\phi_1^2\phi_2 + \phi_4) = d_x[R^{n-1} - (\int M\phi_2)(\int M^{n-1}\phi_2) + R'] - \phi_1 M^n\phi_2 \ . \tag{4.3.31}$$

The above reasoning also gives R' as composed of polynomials in ϕ_1 and its x-derivatives. Using equation (4.3.26), equation (4.3.31) becomes

$$\phi_1 M^{n-1}(\phi_1^2\phi_2 + \phi_4) = d_x W^{n-1} \tag{4.3.32}$$

where

$$2W^{n-1} \equiv R^{n-1} - (\int M\phi_2)(\int M^{n-1}\phi_2) + R' \ , \tag{4.3.33}$$

that is, W^{n-1} is a polynomial in ϕ_1 and its x-derivatives.

Since

$$M^n \equiv d_x[M_x^{n-1} + \tfrac{2}{3}\phi_1 \int \phi_1 M^{n-1}] \ , \tag{4.3.34}$$

the assumption that M^{n-1} is a polynomial, and equation (4.3.32), imply that M^n is the x-derivative of a polynomial, and hence is itself a polynomial. Explicit calculation gives

$$M(\phi_1^2\phi_2 + \phi_4) = \frac{5}{6}\,\phi_1^4\phi_2 + \frac{5}{3}\,\phi_2^3 + \frac{20}{3}\,\phi_1\phi_2\phi_3 + \frac{5}{3}\,\phi_1^2\phi_4 \quad . \tag{4.3.35}$$

The inductive process is started by equation (4.3.35), and the expression

$$M^n(\phi_1^2\phi_2 + \phi_4)$$

is proved a polynomial in ϕ_1 and its x-derivatives for all positive n. This expression is actually proved to be the x-divergence of such a polynomial.

Equation (4.3.22) will be proved by deriving a Lagrangian for the Euler-Lagrange equation

$$M^n(\phi_2) \equiv M^{n-1}(\phi_1^2\phi_2 + \phi_4) = 0 \quad . \tag{4.3.36}$$

Equation (4.3.36) is of no interest in itself, but if a Lagrangian can be derived for it, Noether's theorem can be used to derive equation (4.3.22). Note that since Noether's theorem is in terms of general field variables ϕ, this proof is not restricted to solutions of equation (4.3.36).

The problem of finding a Lagrangian, given the equation of motion, is also known as the *inverse problem of the calculus of variations*. The work of Atherton and Homsy (1975) on this problem, which is presented briefly in appendix J, will be used here to derive a Lagrangian for equation (4.3.36).

Atherton and Homsy (1975) show that equation (4.3.36) has a Lagrangian if and only if *the Frechet derivative of the left-hand side exists and is symmetrical.* Since the left-hand side has been proved to be a polynomial in ϕ_1 and its x-derivatives, in the preceding lemma, its Frechet derivative exists and is given by the formula

$$[M^n(\phi_2)]'_\phi(\sigma) = \sum_a \frac{\partial M^n}{\partial \phi_a}(\phi_2)\sigma_a \quad , \tag{4.3.37}$$

where

$$\phi_a \equiv \phi_{\mu_1 \ldots \mu_a} \quad ,$$

$$\sigma_a \equiv \sigma_{\mu_1 \ldots \mu_a} \quad ,$$

and σ is an arbitrary test function of ϕ and its derivatives.

Symmetry of the Frechet derivative requires that

$$\psi[M^n(\phi_2)]'_\phi(\sigma) = \sigma[M^n(\phi_2)]'_\phi(\psi) + \frac{dV}{dx} \quad , \tag{4.3.38}$$

for arbitrary functions ψ,σ of the field variable, where V is any function which vanishes with ϕ and its derivatives. Equation (4.3.38) will be written

$$\boxed{\psi[M^n(\phi_2)]'_\phi(\sigma) \triangleq \sigma[M^n(\phi_2)]'_\phi(\psi) \quad ,} \tag{4.3.39}$$

where $\triangleq$ means *equals within the* x-*divergence of such a fucntion* V. This equation (4.3.39) must be proved to be true, in order that a Lagrangian exists for equation (4.3.36).

For simpler notation in the proof of equation (4.3.39), define

$$F^k \equiv M^k(\phi_2) \quad , \tag{4.3.40}$$

for all k. Then

$$F^{k+1} = M\, F^k \quad . \tag{4.3.41}$$

Equation (4.3.39) will be proved by induction. Assume that

$$\psi(F^i)'_\phi(\sigma) \triangleq \sigma(F^i)'_\phi(\psi) \tag{4.3.42}$$

for

$$i = 0, 1, \ldots n \quad .$$

Then

$$\psi(F^{n+1})'_\phi(\sigma) = \psi(MF^n)'_\phi(\sigma) \quad , \tag{4.3.43}$$

$$= \psi M(F^n)'_\phi(\sigma) + \psi M'_\phi(\sigma)(F^n), \tag{4.3.44}$$

where

$$M'_\phi(\sigma)(F^n) = \frac{4}{3}\, \phi_1\sigma_1 F^n + \frac{2}{3}\, \sigma_2 \int \phi_1 F^n + \frac{2}{3}\, \phi_2 \int \sigma_1 F^n \quad . \tag{4.3.45}$$

The first term on the right-hand side of equation (4.3.44) can be expanded as in equation (4.3.16), to give

$$\psi M(F^n)'_\phi(\sigma) \triangleq (\int M\psi_1)(F^n)'_\phi(\sigma) - \frac{2}{3}\, \phi_1\psi_1 \int \phi_1(F^n)'_\phi(\sigma)$$
$$- \frac{2}{3}\, (\int \phi_1\psi_1)\phi_1(F^n)'_\phi(\sigma) \quad . \tag{4.3.46}$$

The last two terms in this equation cannot be thrown out as an x-derivative, because they are the x-derivative of a term which does not vanish with ϕ and its derivatives. However, they can be manipulated as follows:

$$- \frac{2}{3}\,\phi_1\psi_1 \int \phi_1 (F^n)'_\phi(\sigma) \;-\; \frac{2}{3}\Big(\int \phi_1\psi_1\Big)\phi_1 (F^n)'_\phi(\sigma)$$

$$= - \frac{2}{3}\,\phi_1\psi_1 \int (\phi_1 F^n)'_\phi(\sigma) \;-\; \frac{2}{3}\Big(\int \phi_1\psi_1\Big)(\phi_1 F^n)'_\phi(\sigma)$$

$$+ \frac{2}{3}\,\phi_1\psi_1 \int \sigma_1 F^n + \frac{2}{3}\Big(\int \phi_1\psi_1\Big)\sigma_1 F^n \quad , \tag{4.3.47}$$

$$= \frac{d}{dx}\Big[- \frac{2}{3}\Big(\int \phi_1\psi_1\Big)\Big(\int \phi_1 F^n\Big)'_\phi(\sigma)\Big]$$

$$+ \frac{2}{3}\,\phi_1\psi_1 \int \sigma_1 F^n + \frac{2}{3}\,\sigma_1 F^n \int \phi_1\psi_1 \quad . \tag{4.3.48}$$

It was proved at equation (4.3.32) that

$$\int \phi_1 F^n$$

is a polynomial in ϕ_1 and its x-derivatives. Hence its Frechet derivative will also be such a polynomial, and the x-divergence term in equation (4.3.48) vanishes with ϕ and its derivatives. Equation (4.3.46) can thus be written

$$\psi M (F^n)'_\phi(\sigma) \doteq \Big(\int M\psi_1\Big)(F^n)'_\phi(\sigma) + \frac{2}{3}\,\phi_1\psi_1 \int \sigma_1 F^n$$

$$+ \frac{2}{3}\,\sigma_1 F^n \int \phi_1\psi_1 \quad . \tag{4.2.49}$$

The last term on the right-hand side of equation (4.3.44) is

$$\psi\, M_{\phi}^{'}(\sigma)(F^{n}) \hat{=} -\frac{2}{3}\,\psi_1\sigma_1 \int \phi_1 F^{n} - \frac{2}{3}\,\phi_1\psi_1 \int \sigma_1 F^{n} \quad , \tag{4.3.50}$$

so that equation (4.3.44) can be written as

$$\psi(F^{n+1})_{\phi}^{'}(\sigma) \hat{=} \left(\int M\psi_1\right)(F^{n})_{\phi}^{'}(\sigma) + \frac{2}{3}\,\sigma_1 F^{n} \int \phi_1\psi_1$$

$$- \frac{2}{3}\,\psi_1\sigma_1 \int \phi_1 F^{n} \quad . \tag{4.3.51}$$

Since F^n has a symmetrical Frechet derivative by assumption, the first term on the right-hand side of equation (4.3.51) is

$$\left(\int M\psi_1\right)(F^{n})_{\phi}^{'}(\sigma) \hat{=} \sigma(F^{n})_{\phi}^{'}\left(\int M\psi_1\right) \quad . \tag{4.3.52}$$

Applying equation (4.3.51) to the term on the right-hand side of equation (4.3.52), it becomes

$$\sigma(F^{n})_{\phi}^{'}\left(\int M\psi_1\right) \hat{=} \left(\int M\sigma_1\right)(F^{n-1})_{\phi}^{'}\left(\int M\psi_1\right)$$

$$+ \frac{2}{3}\,(M\psi_1)F^{n-1} \int \phi_1\sigma_1 - \frac{2}{3}\,\sigma_1(M\psi_1)\int \phi_1 F^{n-1} \quad . \tag{4.3.53}$$

Hence equation (4.3.51) becomes

$$\psi(F^{n+1})_{\phi}^{'}(\sigma) \hat{=} \left(\int M\sigma_1\right)(F^{n-1})_{\phi}^{'}\left(\int M\psi_1\right) + \frac{2}{3}\,\sigma_1 F^{n} \int \phi_1\psi_1$$

$$- \frac{2}{3}\,\psi_1\sigma_1 \int \phi_1 F^{n} + \frac{2}{3}(M\psi_1)F^{n-1} \int \phi_1\sigma_1$$

$$- \frac{2}{3}\,\sigma_1(M\psi_1) \int \phi_1 F^{n-1} \quad . \tag{4.3.54}$$

Since the Frechet derivative of F^{n-1} is symmetric by assumption, the first term on the right-hand side of equation (4.3.54) is symmetric in σ and ψ, as also is the third term by inspection. The remaining terms are not obviously symmetric, but will be shown to be so:

$$\frac{2}{3}\sigma_1 F^n \int \phi_1\psi_1 + \frac{2}{3}(M\psi_1)F^{n-1}\int \phi_1\sigma_1 - \frac{2}{3}\sigma_1(M\psi_1)\int \phi_1 F^{n-1}$$

$$= \frac{2}{3}\sigma_1[(d_x^2 + \frac{2}{3}\phi_1^2 + \frac{2}{3}\phi_2\int \phi_1)F^{n-1}]\int \phi_1\psi_1$$

$$+ \frac{2}{3}[(d_x^2 + \frac{2}{3}\phi_1^2 + \frac{2}{3}\phi_2\int \phi_1)\psi_1]F^{n-1}\int \phi_1\sigma_1$$

$$- \frac{2}{3}\sigma_1[(d_x^2 + \frac{2}{3}\phi_1^2 + \frac{2}{3}\phi_2\int \phi_1)\psi_1]\int \phi_1 F^{n-1}, \tag{4.3.55}$$

$$= \frac{2}{3}\frac{d}{dx}[-\sigma_1\psi_2\int \phi_1 F^{n-1} + \sigma_1 F_x^{n-1}\int \phi_1\psi_1 - \sigma_2 F^{n-1}\int \phi_1\psi_1]$$

$$+ \frac{2}{3}\sigma_2\psi_2\int \phi_1 F^{n-1} + \frac{2}{3}\phi_1 F^{n-1}(\sigma_1\psi_2 + \psi_1\sigma_2)$$

$$+ \frac{2}{3}F^{n-1}(\sigma_3\int \phi_1\psi_1 + \psi_3\int \phi_1\sigma_1) - \frac{2}{3}\sigma_1\psi_1\phi F_x^{n-1}$$

$$- \frac{4}{9}\psi_1\sigma_1\phi_1^2\int \phi_1 F^{n-1} + \frac{4}{9}\phi_2 F^{n-1}(\int \phi_1\psi_1)(\int \phi_1\sigma_1)$$

$$+ \frac{4}{9}\phi_1^2 F^{n-1}(\psi_1\int \phi_1\sigma_1 + \sigma_1\int \phi_1\psi_1) . \tag{4.3.56}$$

The terms on the right-hand side of equation (4.3.56) are either x-derivatives or symmetric in σ and ψ. Hence the Frechet derivative of F^{n+1} is symmetric if the Frechet derivatives of F^i for $i < n+1$ are symmetric. Since

$$\psi(F^0)'_\phi(\sigma) = \psi(\phi_2)'_\phi(\sigma)$$

$$= \psi\sigma_2$$

$$\hat{=} -\psi_1\sigma_1 \quad , \tag{4.3.57}$$

$$\psi(F^1)'_\phi(\sigma) = \psi(\phi_1^2\phi_2 + \phi_4)'_\phi(\sigma)$$

$$= \psi(2\phi_1\phi_2\sigma_1 + \phi_1^2\sigma_2 + \sigma_4)$$

$$\hat{=} -\phi_1^2\psi_1\sigma_1 + \psi_2\sigma_2 \quad , \tag{4.3.58}$$

the inductive process is started, and *the Frechet derivative of* F^i *is symmetric for all* i.

The work of Atherton and Homsy (1975) gives the Lagrangian of F^n as

$$\boxed{L^n = \phi \int_0^1 F^n(\lambda\phi)d\lambda \ .} \tag{4.3.59}$$

Since F^n is a polynomial in ϕ_1 and its x-derivatives, L^n is a polynomial in ϕ and its x-derivatives. Hence L^n is not explicitly dependent on time, and the energy conservation relation is [see appendices B and C, and Noether's relation (2.0.4)]:

$$-\phi_t F^n = d_\mu[\pi^\mu(L^n)\phi_t - L^n\delta_\mu^t] \quad . \tag{4.3.60}$$

L^n doesn't depend on time-derivatives of ϕ, so equation (4.3.60) is

$$\phi_t M^{n-1}(\phi_1^2\phi_2 + \phi_4) = d_t(L^n) + d_x[-\pi^x(L^n)\phi_t] \quad . \tag{4.3.61}$$

This completes the proof of the partial differential part, with the identification from equation (4.3.22):

$$T_n \equiv L^{n+1},$$

$$X_n' \equiv -\pi^x(L^{n+1})\phi_t \quad .$$

(4.3.ii) THE INTEGRO-DIFFERENTIAL PART

In this section, equation (4.3.10) will be proved:

$$\boxed{\phi_t M^n(\phi_{1t}) = d_x Q^n \quad ,} \tag{4.3.62}$$

where Q^n is equivalent to a function which vanishes with ϕ and its derivatives. Relation (4.3.11) will be used to prove this:

$$fM^n(g_x) = d_x R^n(f,g) - g\, M^n(f_x) \tag{4.3.63}$$

with

$$g = f = \phi_t \quad .$$

The first requirement for R^n to be equivalent to a function which vanishes with ϕ and its derivatives is [equation (4.3.12)]:

$$M^i(\phi_{1t}) \approx d_x G^i \tag{4.3.64}$$

for $i < n$, where G^i vanishes with ϕ and its derivatives. The form of M ensures that this holds:

$$M^i(\phi_{1t}) = d_x[(d_x + \frac{2}{3}\phi_1 \int \phi_1)M^{i-1}(\phi_{1t})] \quad . \tag{4.3.65}$$

The second requirement is [equation (4.3.13)]:

$$\phi_1 M^i(\phi_{1t}) \approx d_x F^i \quad , \tag{4.3.66}$$

where F^i vanishes with ϕ and its derivatives, for $i < n$. This is satisfied as follows:

$$\begin{aligned} \phi_1 M^i(\phi_{1t}) &= \phi_1 M^i(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) - \phi_1 M^i(\phi_1^2\phi_2 + \phi_4) , \\ &\approx - \phi_1 M^i(\phi_1^2\phi_2 + \phi_4) \quad , \end{aligned} \tag{4.3.67}$$

remembering that $\approx$ means *equivalent to*, as in appendix D, that is, equal within a trivially conserved vector. At equation (4.3.28) it was proved that the term on the right-hand side of equation (4.3.67) is equal to the x-derivative of a polynomial in ϕ_1 and its x-derivatives. Hence the requirement (4.3.66) is met.

Relation (4.3.63) is

$$\phi_t M^n(\phi_{1t}) = d_x R^n - \phi_t M^n(\phi_{1t}) \quad , \tag{4.3.68}$$

that is,

$$\phi_t M^n(\phi_{1t}) = d_x Q^n \quad , \tag{4.3.69}$$

where

$$Q^n \equiv \frac{1}{2} R^n \quad , \tag{4.3.70}$$

and equation (4.3.62) is proved.

Adding together equations (4.3.61) and (4.3.62) gives

$$\boxed{\begin{aligned} \phi_t M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) &= d_t(\,L^{n+1}) \\ &+ d_x[-\pi^x(L^{n+1})\phi_t + Q^n] \quad , \end{aligned}} \tag{4.3.71}$$

that is, L^{n+1} is a conserved energy density for the n^{th} higher-order equation (4.3.7):

$$M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) = 0 \quad . \tag{4.3.72}$$

From the equation (4.3.59) for L^n, these energy densities are linearly independent of each other, and are polynomials in ϕ and its x-derivatives. Since solutions to the modified KdV equation,

$$\phi_{1t} + \phi_1^2\phi_2 + \phi_4 = 0 \quad , \tag{4.3.73}$$

must solve equations (4.3.72) for all positive n, the L^n must form an infinite set of polynomial conserved densities for the modified KdV equation. Since the set of polynomial conserved densities for the modified KdV equation is shown to be unique at equation (4.2.11), the set of densities L^n

must be equivalent to that found by Miura, Gardner and Kruskal (1968).

(4.4) CONSERVED DENSITIES OF THE MODIFIED KDV EQUATION IDENTIFIED AS MOMENTUM DENSITIES

The work of section (4.3) also provides a basis for identification of the infinite set of polynomial conserved densities of the modified KdV equation as *momentum densities* of the higher-order equations

$$M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) = 0 \quad , \qquad n = 0,1, \ldots \tag{4.4.1}$$

To show this, it is necessary and sufficient to prove that

$$\boxed{\phi_x M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_4) = d_t Y_n + d_x X_n} \tag{4.4.2}$$

for each n. The *partial differential* part of equation (4.4.2) has already been proved, in equation (4.3.32):

$$\boxed{\phi_1 M^n(\phi_1^2\phi_2 + \phi_4) = d_x W^n \quad ,} \tag{4.4.3}$$

where W^n is a polynomial in ϕ_1 and its x-derivatives.

The remaining *integro-differential* part can be proved by recalling relation (4.3.11), whose requirements have been shown to be met for this case at equation (4.3.67):

$$\phi_1 M^n(\phi_{1t}) = d_x R^n - \phi_t M^n(\phi_2) \quad . \tag{4.4.4}$$

Recalling that

$$M(\phi_2) = \phi_1^2\phi_2 + \phi_4 \ , \tag{4.4.5}$$

result (4.3.61) can be written

$$\phi_t M^n(\phi_2) = d_t(\ L^n) + d_x[-\pi^x(L^n)\phi_t] \quad . \tag{4.4.6}$$

Substituting this into equation (4.4.4) gives

$$\boxed{\phi_1 M^n(\phi_{1t}) = d_t(-L^n) + d_x[R^n + \pi^x(L^n)\phi_t] \ ,} \tag{4.4.7}$$

which is the integro-differential part.

Combining equations (4.4.3) and (4.4.7) gives

$$\phi_1 M^n(\phi_{1t} + \phi_1^2\phi_2 + \phi_3) = d_t(-L^n) + d_x[R^n + \pi^x(L^n)\phi_t + W^n] \ . \tag{4.4.8}$$

This completes the proof of equation (4.4.2), with

$$Y_n = -L^n \ ,$$

$$X_n = R^n + \pi^x(L^n)\phi_t + W^n \ .$$

Hence the momentum densities, which are the polynomials $-L^n$, are conserved for solutions to equations (4.4.1), and in particular are conserved for solutions to the modified KdV equation.

Note that each momentum density is identical to the energy density of the preceding equation of motion (4.4.1), found in section (4.3). *Hence the polynomial conserved densities of the modified* KdV *equation can be identified either as energies or as momenta of the higher-order equations* (4.4.1) [within a negative sign].

CHAPTER V

THE SINE-GORDON EQUATION

The technique used in chapters (3) and (4) on the conservation laws of the KdV and the modified KdV equations is applied in this chapter to the infinite set of polynomial conserved densities found for the *Sine-Gordon equation* [see appendix K for references and examples of these densities]. Steudel (1975a) has also associated these conservation laws with invariance under infinitesimal extended Bäcklund transformations, in the same way as for the KdV and the modified KdV equations. His work on the Sine-Gordon equation will not be presented in this thesis, since the technique has already been described.

The form of the Sine-Gordon equation used will be

$$\boxed{\phi_{xt} - \sin(\phi) = 0 \quad .} \tag{5.0.1}$$

Note that equation (5.0.1) is not of the same type as the KdV and modified KdV equations, that is,

$$\phi_{xt} - K(\phi_x) = 0 \quad , \tag{5.0.2}$$

where K is a nonlinear differential operator in x, operating on ϕ_x.

Hence it is not surprising that equation (5.0.1) behaves somewhat differently to equation (5.0.2) when one attempts to find an infinite set of higher-order equations with conserved energy-momentum densities equivalent to the infinite set of densities for equation (5.0.1).

In particular, the energy densities of the enveloping higher-order equations are found to be zero, and the momentum densities are found to be linearly independent polynomials in ϕ_1 and its x-derivatives. Since the first three of these momentum densities are found to be equivalent to the first three of the infinite set of conserved densities found by Sanuki and Konno (1974) for the Sine-Gordon equation, it is highly likely that the two sets of densities are equivalent. A rigorous proof of this equivalence is not attempted in this thesis.

For the Sine-Gordon equation, the choice of an operator which is analogous to the operators H and M of chapters (3) and (4) is not obvious. Operator H was already present in the literature on the KdV equation, and the property that the energy densities of the equations

$$H^n(\phi_{1t} + \phi_1\phi_2 + \phi_4) = 0 \ , \quad n = 0,1,2 \ldots \qquad (5.0.3)$$

are conserved polynomials, was discovered with the operator H already defined. Operator M for the modified KdV equation was found by trial and error, using the form of H as a guide, and the fact that

$$H(\phi_2) = \phi_1\phi_2 + \phi_4 \qquad (5.0.4)$$

as a further guide. [Operator M could have been derived from operator H by using the transformation discovered by Miura (1967) relating the modified KdV equation to the KdV equation, as in the paper by Olver (1976)]. The operator for the Sine-Gordon equation was found by noting that Olver (1976) had used operators H and M as *recursion operators*, generating an infinite series of infinitesimal *higher-order symmetries* for the KdV and modified KdV equations, respectively. Olver exhibits two operators for the Sine-Gordon equation [my notation]:

$$\boxed{\begin{aligned} S &\equiv d_x^2 + \phi_1^2 + \phi_2 \int \phi_1 \,, \\ G &\equiv d_x^2 + \phi_1^2 - \phi_1 \int \phi_2 \,. \end{aligned}} \tag{5.0.5}$$

Operator S is used by Olver to show that operator G is a recursion operator for the Sine-Gordon equation. Both operators will be used in the following sections of this chapter. In particular, it will be shown that the operator S is the desired generator of higher-order Sine-Gordon equations.

(5.1) <u>THE CONSERVED DENSITIES OF THE SINE-GORDON EQUATION IDENTIFIED AS MOMENTUM DENSITIES</u>

In this section it will be proved that

$$\boxed{\phi_1 S^n(\phi_{1t} - \sin\phi) = d_t\hat{T}^n + d_x X^n, \quad n = 0,1,\ \ldots} \tag{5.1.1}$$

where $\hat{T}^n$ is a polynomial in ϕ_1 and its x-derivatives, and where X^n vanishes with ϕ and its derivatives. Equation (5.1.1) gives the momentum density $\hat{T}^n$ of the higher-order Sine-Gordon equation

$$S^n(\phi_{1t} - \sin\phi) = 0 \quad , \tag{5.1.2}$$

conserved for solutions to equation (5.1.2), and hence conserved for solutions to the Sine-Gordon equation (5.0.1). If the $\hat{T}^n$ are linearly independent, an infinite set of polynomial conserved densities has been generated by equation (5.1.1) for the Sine-Gordon equation, and for the higher-order equations (5.1.2).

It should be noted that operator S annihilates the sine term in the Sine-Gordon equation,

$$\begin{aligned} S(\sin\phi) &= (d_x^2 + \phi_1^2 + \phi_2 \int \phi_1)\sin\phi \quad , \\ &= \phi_2\cos\phi - \phi_1^2\sin\phi + \phi_1^2\sin\phi - \phi_2\cos\phi \quad , \\ &= 0 \quad . \end{aligned} \tag{5.1.3}$$

As will be seen in section (5.2), this is the reason that the energy densities of equations (5.1.2) are zero, and it is clear that only the *integro-differential* part of equation (5.1.1) need be proved, that is

$$\boxed{\phi_1 S^m(\phi_{1t}) = d_t\hat{T}^m + d_x X^m \quad , \quad m > 0 \quad .} \tag{5.1.4}$$

The higher-order Sine-Gordon equations (5.1.2) become

$$S^m(\phi_{1t}) = 0 \quad , \quad m > 0 \quad . \tag{5.1.5}$$

The following lemmas will be useful in proving equations (5.1.4):

Lemma (5.1): That

$$\boxed{fS^n(g) = gG^n(f) + d_x Q^n(f,g) \quad ,} \tag{5.1.6}$$

where Q^n is *equivalent* to a function which vanishes with ϕ and its derivatives, in the sense of appendix D, whenever the functions f and g of ϕ and its derivatives satisfy the requirements that either

$$\phi_2 G^k(f) \approx d_x L^k(\phi, \phi_\mu, \; \ldots \;) \tag{5.1.7}$$

or

$$\phi_1 S^{k-1}(g) \approx d_x N^k(\phi, \phi_\mu, \; \ldots \;)$$

for all $k \leqslant n$, where the functions L^k and N^k vanish with their arguments. [Recall that $\approx$ means *is equivalent to* in the senseof appendix D].

Proof:

$$fS^n(g) = f(d_x^2 + \phi_1^2 + \phi_2 \int \phi_1)S^{n-1}(g) \quad , \tag{5.1.8}$$

$$= d_x[fd_x S^{n-1}(g) - f_x S^{n-1}(g) + (\int \phi_2 f)\int \phi_1 S^{n-1}(g)]$$

$$+ G(f)S^{n-1}(g) \quad , \tag{5.1.9}$$

$$= G(f)S^{n-1}(g) + d_x Q' \quad , \tag{5.1.10}$$

where Q' is equivalent to an acceptable function if either

$$\phi_2 f \approx d_x L^0(\phi, \phi_\mu, \ldots)$$

or (5.1.11)

$$\phi_1 S^{n-1}(g) \approx d_x N^{n-1}(\phi, \phi_\mu, \ldots) \quad ,$$

where the functions L^0 and N^{n-1} vanish with their arguments.

Applying the result (5.1.10) to the first term on the right-hand side of equation (5.1.10), and repeating this process n times altogether, gives the proof of lemma (5.1), with

$$Q^n(f,g) = \sum_{i=0}^{n-1} \{G^i(f) d_x S^{n-i-1}(g) - S^{n-i-1}(g) d_x G^i(f)$$

$$+ [\int \phi_2 G^i(f)][\int \phi_1 S^{n-i-1}(g)]\} \quad . \tag{5.1.12}$$

Lemma (5.2): That

$$\boxed{f_x G^m(g) = d_x R^m(f,g) - g_x G^m(f) \quad ,} \tag{5.1.13}$$

where $R^m(f,g)$ is equivalent to a function which vanishes with ϕ and its derivatives, whenever the functions f and g satisfy the requirements that either

$$\phi_2 G^r(f) \approx d_x B^r(\phi, \phi_\mu, \ldots)$$

or (5.1.14)

$$\phi_2 G^r(g) \approx d_x C^r(\phi, \phi_\mu, \ldots) \quad ,$$

for all $r < m$, where the functions B^n and C^n vanish with their arguments.

Proof:

$$f_x G^m(g) = f_x(d_x^2 + \phi_1^2 - \phi_1 \int \phi_2) G^{m-1}(g) \quad , \tag{5.1.15}$$

$$= d_x\{f_1 d_x G^{m-1}(g) - f_2 G^{m-1}(g) - \phi_1 f \int \phi_2 G^{m-1}(g)$$

$$+ [\int \phi_2 f][\int \phi_2 G^{m-1}(g)]\} + G^{m-1}(g) d_x G(f), \tag{5.1.16}$$

$$= d_x R' + G^{m-1}(g)\; d_x G(f) \quad , \tag{5.1.17}$$

where R' is an acceptable flux term provided that either

$$\phi_2 f \approx d_x B^0(\phi, \phi_\mu, \ldots)$$

or

$$\phi_2 G^{m-1}(g) \approx d_x C^{m-1}(\phi, \phi_\mu, \ldots) \quad , \tag{5.1.18}$$

where B^0 and C^{m-1} vanish with their arguments.

Applying the result (5.1.17) to the last term on the right-hand side of equation (5.1.17), and repeating this process *m* times altogether, gives the proof of lemma (5.2), with

$$R^m(f,g) = \sum_{i=0}^{m-1} \{d_x G^i(f) d_x G^{m-i-1}(g) - d_x^2 G^i(f)\; G^{m-i-1}(g)$$

$$- \phi_1 G^i(f) \int \phi_2\; G^{m-i-1}(g) + [\int \phi_2 G^i(f)][\int \phi_2 G^{m-i-1}(g)]\}$$

$$+ g G^m(f). \tag{5.1.19}$$

Using lemma (5.1), the left-hand side of equation (5.1.4) is

$$\boxed{\phi_1 S^m(\phi_{1t}) = \phi_{1t} G^m(\phi_1) + d_x Q^m(\phi_1, \phi_{1t}) \quad ,} \tag{5.1.20}$$

where Q^m is an acceptable flux term if either

$$\phi_2 G^k(\phi_1) \approx d_x L^k(\phi, \phi_\mu, \ \ldots\ )$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5.1.21)

$$\phi_1 S^{k-1}(\phi_t) \approx d_x N^{k-1}(\phi, \phi_\mu, \ \ldots) \quad ,$$

for all $k \leqslant m$ where L^k and N^k vanish with their arguments.

To satisfy requirement (5.1.21), it will be shown that

$$\phi_2 G^k(\phi_1) = d_x P^k \quad , \tag{5.1.22}$$

where P^k is a polynomial in ϕ_1 and its x-derivatives, for all k. The proof is by induction as follows: assume that equation (5.1.22) holds for all $k < n$. Lemma (5.2) gives

$$\phi_2 G^n(\phi_1) = -\phi_2 G^n(\phi_1) + d_x R^n(\phi_1, \phi_1) \ , \tag{5.1.23}$$

where from equation (5.1.19),

$$R^n(\phi_1,\phi_1) = \sum_{i=0}^{n-1} \{d_x G^i(\phi_1) d_x G^{n-i-1}(\phi_1) - d_x^2 G^i(\phi_1)\ G^{n-i-1}(\phi_1)$$

$$- \phi_1 G^i(\phi_1) \int \phi_2 G^{n-i-1}(\phi_1) + [\int \phi_2\ G^i(\phi_1)][\int \phi_2 G^{n-i-1}(\phi_1)]\}$$

$$+ \phi_1 G^n(\phi_1) \quad . \tag{5.1.24}$$

The inductive assumption (5.1.22) implies that

$$R^n(\phi_1 \phi_1)$$

is a polynomial in ϕ_1 and its x-derivatives, and hence from equation (5.1.23),

$$\phi_2 G^n(\phi_1) = d_x\{\tfrac{1}{2}R^n(\phi_1,\phi_1)\} = d_x P^n \quad . \tag{5.1.25}$$

Since explicit calculation gives

$$\phi_2 G^0(\phi_1) = \phi_2\phi_1 = d_x(\tfrac{1}{2}\phi_1^2) \tag{5.1.26}$$

and

$$\phi_2 G^1(\phi_1) = d_x(\tfrac{1}{8}\phi_1^4 + \tfrac{1}{2}\phi_2^2) \quad , \tag{5.1.27}$$

the inductive process is started, and equation (5.1.22) is proved for all k.

As a corollary to this proof, since

$$G^m(\phi_1) = (d_x^2 + \phi_1^2 - \phi_1 \int \phi_2) G^{m-1}(\phi_1) \quad , \tag{5.1.28}$$

it is clear that if

$$G^{m-1}(\phi_1)$$

is a polynomial, then result (5.1.22) implies that

$$G^m(\phi_1)$$

is a polynomial. Since

$$G^1(\phi_1) = \phi_3 + \frac{1}{2}\phi_1^2 \quad , \tag{5.1.29}$$

the induction is started, and

$$G^m(\phi_1)$$

is proved a polynomial in ϕ_1 and its x-derivatives for all m.

To prove that equation (5.1.4) holds, equation (5.1.20) implies that it is required to prove that

$$\boxed{\phi_{1t}G^n(\phi_1) = d_t\hat{T}^n + d_xX'^n \quad .} \tag{5.1.30}$$

Transform the dependent variables,

$$u \equiv \phi_x \quad . \tag{5.1.31}$$

Equation (5.1.30) becomes

$$\boxed{u_tG^n(u) = d_t\hat{T}^n + d_xX'^n \quad .} \tag{5.1.32}$$

Equation (5.1.32) will be proved by deriving a Lagrangian for the equation

$$G^n(u) = 0 \quad , \tag{5.1.33}$$

using the theory in appendix J for solving the inverse problem of the calculus of variations [see also section (4.3.i) for a previous application of this technique].

If equation (5.1.33) is to have a Lagrangian, the results presented in appendix J require that the Frechet derivative of $G^n(u)$ be symmetrical, that is

$$\boxed{\sigma[G^n(u)]'_u(\psi) \triangleq \psi[G^n(u)]'_u(\sigma) \quad ,} \tag{5.1.34}$$

where σ,ψ are arbitrary functions of ϕ, and where, as in section (4.3.i), $\triangleq$ means *equals within the* x-*derivative of any function which vanishes with* u *and its derivatives.* Equation (5.1.34) will be proved to be true for all n by induction, assuming that

$$\sigma[G^i(u)]'_u(\psi) \triangleq \psi[G^i(u)]'_u(\sigma) \tag{5.1.35}$$

for all $i < n$.

For simplicity of notation in the remainder of this section define

$$T^k \equiv G^k(u), \quad k = 0, 1, \ldots \tag{5.1.36}$$

The left-hand side of equation (5.1.34) is

$$\sigma(T^n)'_u(\psi) = \sigma(G)'_u(\psi)(T^{n-1}) + \sigma G(T^{n-1})'_u(\psi) , \tag{5.1.37}$$

where

$$(G)'_u(\psi)(T^{n-1}) \equiv (2u\psi - u\int\psi_1 - \psi\int u_1)T^{n-1} \quad . \tag{5.1.38}$$

The last term in equation (5.1.37) can be rearranged as

$$\sigma G(T^{n-1})'_u(\psi) = \sigma(d_x^2 + u^2 - u\int u_1)(T^{n-1})'_u(\psi) \quad , \tag{5.1.39}$$

$$= d_x[\sigma d_x(T^{n-1})'_u(\psi) - \sigma_1(T^{n-1})'_u(\psi)] + \sigma_2(T^{n-1})'_u(\psi)$$

$$+ u^2\sigma(T^{n-1})'_u(\psi) - \sigma u\int(u_1T^{n-1})'_u(\psi) + \sigma u\int\psi_1T^{n-1}, \tag{5.1.40}$$

$$\hat{=} d_x[(-\int\sigma u)(\int u_1T^{n-1})'_u(\psi)] + \psi_1T^{n-1}\int\sigma u$$

$$+ \sigma u\int\psi_1T^{n-1} + S(\sigma)(T^{n-1})'_u(\psi) \quad . \tag{5.1.41}$$

The integral term in the x-derivative in equation (5.1.41) is acceptable since

$$\int u_1T^{n-1} \tag{5.1.42}$$

was proved a polynomial in u and its x-derivatives earlier [see equation (5.1.22) and following work]. Hence equation (5.1.41) becomes

$$\sigma G(T^{n-1})'_u(\psi) \hat{=} \psi_1T^{n-1}\int\sigma u + \sigma u\int\psi_1T^{n-1} + S(\sigma)(T^{n-1})'_u(\psi), \tag{5.1.43}$$

and equation (5.1.37) becomes

$$\sigma(T^n)'_u(\psi) \hat{=} S(\sigma)(T^{n-1})'_u(\psi) + [2u\sigma\psi - \sigma\psi\int u_1 + (\int\sigma u)\psi_1]T^{n-1} \quad . \tag{5.1.44}$$

Assumption (5.1.35) implies that

$$S(\sigma)(T^{n-1})'_u(\psi) \hat{=} \psi(T^{n-1})'_u[S(\sigma)] \quad , \tag{5.1.45}$$

and result (5.1.44) can be applied to the right-hand side of equation (5.1.45) to get

$$\psi(T^{n-1})'_u[S(\sigma)] \triangleq S(\psi)(T^{n-2})'_u[S(\sigma)]$$

$$+ [2u\psi S(\sigma) - \psi S(\sigma)\int u_1 + (\int \psi u)S_x(\sigma)]T^{n-2} \quad . \tag{5.1.46}$$

Substituting from equations (5.1.45) and (5.1.46) into equation (5.1.44) gives

$$\sigma(T^n)'_u(\psi) \triangleq S(\psi)(T^{n-2})'_u[S(\sigma)] + [2u\sigma\psi - \sigma\psi\int u_1 + (\int \sigma u)\psi_1]T^{n-1}$$

$$+ [2u\psi S(\sigma) - \psi S(\sigma)\int u_1 + (\int \psi u)S_x(\sigma)]T^{n-2} \quad . \tag{5.1.47}$$

The first term on the right-hand side of equation (5.1.47) is symmetric by assumption (5.1.35). The only remaining terms which are not obviously symmetric are

$$[2u\psi S(\sigma) - \psi S(\sigma)\int u_1 + (\int \psi u)S_x(\sigma)]T^{n-2} + (\int \sigma u)\psi_1 T^{n-1} \quad . \tag{5.1.48}$$

Using

$$(\int \psi u)S_x(\sigma)T^{n-2} = d_x[(\int \psi u)S(\sigma)T^{n-2}] - (\int \psi u)S(\sigma)d_x T^{n-2}$$

$$- \psi u S(\sigma)T^{n-2} \quad , \tag{5.1.49}$$

and substituting for operators S and G, the expression (5.1.48) becomes

$$u\psi(\sigma_2 + u^2\sigma + u_1\int u\sigma)T^{n-2} - (\sigma_2 + u^2\sigma + u_1\int u\sigma)d_x T^{n-2}\int\psi u$$

$$- \psi(\sigma_2 + u^2\sigma + u_1\int u\sigma)\int u_1 T^{n-2} + \psi_1(\int u\sigma)(d_x^2 + u^2 - u\int u_1)T^{n-2} \,, \qquad (5.1.50)$$

which can be rewritten in the form

$$d_x[u\psi\sigma_1 T^{n-2} + u^2\psi T^{n-2}\int u\sigma - \sigma_1 d_x T^{n-2} \int\psi u - \psi\sigma_1 \int u_1 T^{n-2}$$

$$-u\psi(\int u\sigma) \int u_1 T^{n-2}] + [\sigma_1(\int\psi u) + \psi_1(\int\sigma u)]d_x^2 T^{n-2}$$

$$- [\psi(\int\sigma u) + \sigma(\int\psi u)]u^2 d_x T^{n-2} + \psi_1\sigma_1 \int u_1 T^{n-2}$$

$$- \psi_1\sigma_1\ uT^{n-2} - (\int\sigma u)(\int\psi u)u_1 d_x T^{n-2} \ . \qquad (5.1.51)$$

Clearly, expression (5.1.51) is equivalent to symmetrical terms, so that the right-hand side of equation (5.1.47) is also equivalent to symmetrical terms, and equation (5.1.34) is proved to hold under assumption (5.1.35).

Since

$$\sigma(T)'_u(\psi) = \sigma(u_2 + \tfrac{1}{2}u^3)'_u(\psi) \,,$$

$$= \sigma\psi_2 + \tfrac{3}{2}\sigma u^2\psi \,,$$

$$\triangleq - \sigma_1\psi_1 + \tfrac{3}{2}\sigma\psi u^2 \,, \qquad (5.1.52)$$

which is clearly symmetrical, the inductive process is started, and equation (5.1.34) is proved for all n.

Hence, for the equations

$$G^n(u) = 0 \quad , \qquad n = 1,2, \ldots \tag{5.1.53}$$

the work summarised in appendix J gives the Lagrangians

$$\boxed{L^n = u \int_0^1 T^n(\lambda u)\, d\lambda \quad , \qquad n = 1,2, \ldots} \tag{5.1.54}$$

where λ is some parameter. Clearly, from the properties of G, these Lagrangians are polynomials in u and its x-derivatives, and are linearly independent of each other. Since these Lagrangians do not depend explicitly on time, energy is conserved for each of the equations (5.1.53) [see appendix C], and Noether's relation (2.0.4) gives the energy conservation relations as

$$u_t G^n(u) = d_\mu[-\pi^\mu(L^n)u_t + L^n\delta^t_\mu], \quad n = 1,2, \ldots \tag{5.1.55}$$

Since there are no time-derivatives of u in the Lagrangians, equation (5.1.55) becomes

$$\boxed{u_t G^n(u) = d_t(L^n) + d_x[-\pi^x(L^n)u_t] \quad .} \tag{5.1.56}$$

This completes the proof of equations (5.1.32), with the identification

$$\begin{aligned} \hat{T}^n &= L^n \quad , \\ -X'^n &= \pi^x(L^n)u_t \quad . \end{aligned} \tag{5.1.57}$$

Hence equations (5.1.1) are proved, giving an infinite number of conserved polynomials

$$L^{i},\ i = m,\ m+1,\ \ldots \tag{5.1.58}$$

for each of the equations

$$S^{m}(\phi_{1t} - \sin\phi) = 0,\ m = 0,1,\ \ldots \tag{5.1.59}$$

where each polynomial conserved density L^{m} is identified as a momentum density for each of the higher-order Sine-Gordon equations (5.1.59). Note that the solution set of each higher-order Sine-Gordon equation contains the solution sets of all of those equations of lower order, including the Sine-Gordon equation itself [m = 0], so that conservation laws existing in the higher-order systems must also exist in the lower-order systems [and, in particular, in the Sine-Gordon system].

The infinite set of polynomial conserved densities

$$L^{k},\ k = 0,1,\ \ldots \tag{5.1.60}$$

found in this way for the Sine-Gordon equation may be compared to that set derived by Sanuki and Konno (1974) [see appendix K for the first four of this set]. The first three densities of both sets are found to be equivalent, so that the sets are likely to be equivalent. No rigorous proof of this equivalence [e.g. a uniqueness proof for one or the other set] is known of at present.

(5.2) THE ENERGY DENSITIES OF THE HIGHER-ORDER SINE-GORDON EQUATIONS

The momentum densities of the higher-order equations

$$S^k(\phi_{1t} - \sin\phi) = 0, \quad k = 1,2, \dots \tag{5.2.1}$$

have been found to be linearly independent conserved polynomials in ϕ_1 and its x-derivatives. *In this section the energy densities of equation* (5.2.1) *will be shown to be zero,* as a consequence of the annihilation of the sine term by operator S.

A lemma which will be useful in this proof follows:

Lemma (5.3): That

$$\boxed{fS^n(g_1) = d_x W^n(f,g) - gS^n(f_1) \quad ,} \tag{5.2.2}$$

where $W^n(f,g)$ is equivalent to an acceptable flux term [that is, one which vanishes with ϕ and its derivatives] provided that f and g satisfy the following requirements:

1. either

$$\phi_1 S^k(f_1) \approx d_x D^k(\phi, \phi_\mu, \dots)$$

or (5.2.3)

$$\phi_1 S^{k-1}(g_1) \approx d_x E^{k-1}(\phi, \phi_\mu, \dots) \quad ,$$

for all $k \leqslant n$, where the functions D^k and E^k vanish with their arguments, and

2. either

$$S^k(f_1) \approx d_x M^k(\phi, \phi_\mu, \ldots)$$

or

$$S^k(g_1) \approx d_x N^k(\phi, \phi_\mu, \ldots), \tag{5.2.4}$$

for all $k < n$, where the functions M^k and N^k vanish with their arguments.

Proof:

$$fS^n(g_1) = f(d_x^2 + \phi_1^2 + \phi_2 \int \phi_1) S^{n-1}(g_1) \quad , \tag{5.2.5}$$

$$= d_x\{f d_x S^{n-1}(g_1) - f_1 S^{n-1}(g_1) + f_2 \int S^{n-1}(g_1) + \phi_1 f \int \phi_1 S^{n-1}(g_1)$$

$$- [\int \phi_1 f_1] \int \phi_1 S^{n-1}(g_1) + \phi_1 [\int \phi_1 f_1] \int S^{n-1}(g_1) - [\int S(f_1)] \int S^{n-1}(g_1)\}$$

$$+ [\int S(f_1)] S^{n-1}(g_1) \quad , \tag{5.2.6}$$

$$= d_x W'^n(f,g) + [\int S(f_1)] S^{n-1}(g_1) \quad , \tag{5.2.7}$$

where $W'^n(f,g)$ is an acceptable flux term if

1. either

$$\phi_1 f_1 \approx d_x D^0$$

or

$$\phi_1 S^{n-1}(g_1) \approx d_x E^{n-1} \quad , \tag{5.2.8}$$

with D^k and E^k as above, and

2. either

$$S(f_1) \approx d_x M^1$$

or

$$S^{n-1}(g_1) \approx d_x N^{n-1} \quad , \tag{5.2.9}$$

with M^k and N^k as above.

Applying the result (5.2.7) to the last term in equation (5.2.7), and repeating this process n times altogether, completes the proof of lemma (5.3), with

$$\begin{aligned} W^n(f,g) = \sum_{i=0}^{n-1} \{ [\int S^i(f_1)] d_x S^{n-i-1}(g_1) - S^i(f_1) S^{n-i-1}(g_1) \\ + d_x S^i(f_1) \int S^{n-i-1}(g_1) + \phi_1 [\int S^i(f_1)] \int \phi_1 S^{n-i-1}(g_1) \\ + \phi_1 [\int S^{n-i-1}(g_1)] \int \phi_1 S^i(f_1) - [\int \phi_1 S^i(f_1)] \int \phi_1 S^{n-i-1}(g_1) \\ - [\int S^{i+1}(f_1)] \int S^{n-i-1}(g_1) \} + g \int S^n(f_1) \quad . \end{aligned} \tag{5.2.10}$$

The energy densities of equations (5.2.1) are given by $\bar{T}^k$ in the Noether relation [*if it holds*, in which case they are conserved]

$$\boxed{\phi_t S^k(\phi_{1t} - \sin\phi) = d_t \bar{T}^k + d_x X^k \quad , \qquad k = 1,2, \ldots} \tag{5.2.11}$$

It will be shown that equations (5.2.11) hold with the energy densities identically zero. The left-hand side of equations (5.2.11) is [using lemma (5.3)]

$$\phi_t S^k(\phi_{1t}) = -\phi_t S^k(\phi_{1t}) + d_x W^k(\phi_t, \phi_t), \tag{5.2.12}$$

where the flux term is acceptable provided that

1. $\phi_1 S^i(\phi_{1t}) \approx d_x E^i \quad ,$ (5.2.13)

for all $i < k$, where E^i vanishes with ϕ and its derivatives, and

2. $S^i(\phi_{1t}) \approx d_x M^i \; ,$ (5.2.14)

for all $i < k$, where M^i vanishes with ϕ and its derivatives.

Condition (5.2.13) is met for all $i \geqslant 1$, since

$$\phi_1 S^i(\phi_{1t}) = \phi_1 S^i(\phi_{1t} - \sin\phi) \quad , \quad i \geqslant 1 \; , \tag{5.2.15}$$

which is zero for solutions to the higher-order Sine-Gordon equations (5.2.1) with $k \leqslant i$, and in particular for solutions to the Sine-Gordon equation, that is,

$$\phi_1 S^i(\phi_{1t}) \approx 0 \quad . \tag{5.2.16}$$

Condition (5.2.14) is met because

$$S^i(\phi_{1t}) = d_x[S_x^{i-1}(\phi_{1t}) + \phi_1 \int \phi_1 S^{i-1}(\phi_{1t})] \quad . \tag{5.2.17}$$

Note that by inductive reasoning, the term

$$S_x^{i-1}(\phi_{1t}) \tag{5.2.18}$$

is an acceptable flux term.

Equation (5.2.12) implies that

$$\boxed{\phi_t S^k(\phi_{1t}) = d_x[\tfrac{1}{2} W^k(\phi_t,\phi_t)] \quad ,} \tag{5.2.19}$$

which is equation (5.2.11) with

$$\bar{T}^k = 0 \quad ,$$

$$X^k = \frac{1}{2} W^k(\phi_t,\phi_t) \quad . \tag{5.2.20}$$

This completes the proof that the energy densities of the higher-order Sine-Gordon equation (5.2.1) are zero.

(5.3) A CORRESPONDENCE BETWEEN THE CONSERVED DENSITIES OF THE SINE-GORDON AND THE MODIFIED KDV EQUATIONS

The conserved densities of the modified KdV equation may be obtained by making a simple scale transformation of the field variable in the conserved densities of the Sine-Gordon equation. For example, the second conserved density in appendix K for the Sine-Gordon equation is

$$T_2^S(\phi) = \phi_1\phi_3 + \frac{1}{4}\phi_1^4 \quad , \qquad \phi \equiv u \quad . \tag{5.3.1}$$

Making the scale transformation

$$\phi' = \sqrt{\frac{3}{2}}\,\phi \quad , \tag{5.3.2}$$

equation (5.3.1) becomes

$$T_2^S\left(\sqrt{\frac{2}{3}}\,\phi'\right) = \frac{2}{3}\,\phi_1'\phi_3' + \frac{1}{9}\,\phi_1'^4 \quad , \tag{5.3.3}$$

$$\hat{=} \frac{4}{9}\left(\frac{1}{4}\,\phi_1'^4 - \frac{3}{2}\,\phi_2'^2\right) \quad . \tag{5.3.4}$$

Noting that the second conserved density in appendix I for the modified KdV equation is

$$T_2^M(\phi') = \frac{1}{4}\,\phi_1'^4 - \frac{3}{2}\,\phi_2'^2 \quad , \tag{5.3.5}$$

the densities (5.3.4) and (5.3.5) are clearly equivalent within a constant factor.

A comparison of the operators

$$S = d_x^2 + \phi_1^2 + \phi_2 \int \phi_1 \tag{5.3.6}$$

and

$$M = d_x^2 + \frac{2}{3}\,\phi_1^2 + \frac{2}{3}\,\phi_2 \int \phi_1 \quad , \tag{5.3.7}$$

shows that they are related by the same scale transformation (5.3.2). This relationship, together with the equations [from equation (5.1.4)]

$$\phi_1 S^n(\phi_{1t}) = d_t T_n^S + d_x X_n^S \quad , \tag{5.3.8}$$

and [from equation (4.4.7)]

$$\phi_1 M^n(\phi_{1t}) = d_t T_n^M + d_x X_n^M \quad , \tag{5.3.9}$$

explains the correspondence between the conserved densities of the Sine-Gordon and the modified KdV equations.

CHAPTER VI

THE CLASSICAL NONLINEAR SHALLOW-WATER EQUATIONS

The *irrotational* motion of an inviscid homogeneous fluid under the action of gravity, is described in the long wave approximation by the *classical nonlinear shallow-water equations* [see, for example, Benney (1973)]

$$\boxed{\begin{aligned} &h_t + uh_x + hu_x = 0 \\ &u_t + uu_x + gh_x = 0 \ , \end{aligned}} \tag{6.0.1}$$

where u(x,t) is the horizontal velocity component, h(x,t) is the free surface height, and g is the gravitational acceleration constant.

Benney (1973) and Miura (1974) have derived an infinite set of conservation laws for the case of *non-zero vorticity* [rotational motion], in which there is a dependence on an extra y-coordinate. These conservation laws reduce to an infinite set for the equations (6.0.1) if the motion is required to be irrotational. The first eight polynomial conserved densities obtained in this way for equations (6.0.1) are presented in appendix L.

The technique for identifying the infinite sets of conserved densities for the KdV, the modified KdV, and the Sine-Gordon equations, as energy or momentum densities can also be applied to the classical nonlinear shallow-water equations. This involves a generalisation of the operator generating the higher-order equations to a *matrix* form, and it is found that the results obtained are the natural generalisations of previous results to a vector and matrix formulation. In view of this, the presentation in this chapter is more concise than in previous chapters.

In section (6.1), an infinite set of polynomial conserved *energy* densities is derived for equations (6.0.1). The first four of these densities are found to be equivalent within a constant number to four of the densities presented in appendix L. In section (6.2), the infinite set of densities derived in section (6.1) is given an alternative identification as a set of *momentum* densities.

(6.1) AN INFINITE SET OF CONSERVED ENERGY DENSITIES FOR THE CLASSICAL SHALLOW-WATER EQUATIONS

The equations (6.01) are transformed into a more convenient form by the substitution

$$\phi_{1x} \equiv u, \ \phi_{2x} \equiv h, \ g \equiv 1 \ , \qquad (6.1.1)$$

where the field variable is now the vector

$$\underline{\phi} = (\phi_1, \phi_2) \ . \qquad (6.1.2)$$

The notation

$$\phi_n = \frac{d^n\phi}{dx^n} \tag{6.1.3}$$

is *not* used in this chapter, since no x-derivatives higher than third-order appear. The notation used is, for example,

$$\phi_{1x} \equiv \frac{d\ \phi_1}{dx}\ , \quad \phi_{2x} \equiv \frac{d\ \phi_2}{dx} \quad . \tag{6.1.4}$$

Under the substitution (6.1.1), the classical nonlinear shallow-water equations are

$$\boxed{\underline{F} \equiv \begin{pmatrix} \phi_{2xt} + \phi_{1x}\phi_{2xx} + \phi_{1xx}\phi_{2x} \\ \phi_{1xt} + \phi_{1x}\phi_{1xx} + \phi_{2xx} \end{pmatrix} = \underline{0} \quad .} \tag{6.1.5}$$

A Lagrangian for these equations is

$$L = -\ \phi_{1t}\phi_{2x} - \tfrac{1}{2}\phi_{1x}^2\phi_{2x} - \tfrac{1}{2}\phi_{2x}^2 \quad . \tag{6.1.6}$$

The *higher-order* shallow-water equations are

$$\boxed{\underline{\underline{I}}^n(\underline{F}) = 0\ , \qquad n = 0,1,\ldots} \tag{6.1.7}$$

where

$$\underline{\underline{I}} \equiv \begin{pmatrix} \tfrac{1}{2}\phi_{1x} & , & \phi_{2x} + \tfrac{1}{2}\phi_{2xx}\int^x dx \\ 1 & , & \tfrac{1}{2}\phi_{1x} + \tfrac{1}{2}\phi_{1xx}\int^x dx \end{pmatrix} \quad . \tag{6.1.8}$$

The lower limit of integration is such that $\underline{\phi}$ vanishes at that boundary.

<u>Definition</u>: The *adjoint* $\underline{\phi}^{\dagger}$ of a vector $\underline{\phi}$ is that vector with rows and columns interchanged [all vectors in this chapter are two-dimensional and real]. The adjoint of a matrix is that matrix with rows and columns interchanged. Note that

$$(\underline{\phi}^{\dagger})^{\dagger} = \underline{\phi}, (\underline{\underline{A}}\phi)^{\dagger} = \underline{\underline{A}}^{\dagger}\underline{\phi}^{\dagger} \quad .$$

The left-hand side of equation (6.1.5) can be written

$$\boxed{\underline{F} = \underline{\phi}^{\dagger}_{xt} + \underline{\underline{I}}(\underline{\phi}^{\dagger}_{xx}) \quad .} \qquad (6.1.9)$$

The solution sets of equations (6.1.7) contain that of the shallow-water equations (6.1.5), so that conserved densities for the *higher-order* equations (6.1.7) are also conserved for equations (6.1.5).

It will be shown that

$$\boxed{\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{F}) = d_t T_n + d_x X_n \quad ,} \qquad (6.1.10)$$

where T_n is a polynomial in $\underline{\phi}$ and $\underline{\phi}_x$. Equation (6.1.10) implies, by a natural extension of generalised Noether's theorem to a vector formulation, that T_n is a conserved energy density for the equation of motion

$$\underline{\underline{I}}^n(\underline{F}) = 0 \quad . \qquad (6.1.11)$$

The T_n will be shown to be linearly independent of each other, so that they constitute an infinite set of polynomial conserved densities for the classical nonlinear shallow-water equations (6.1.5).

As in previous chapters, equation (6.1.10) will be proved in two parts, the *partial differential* part

$$\boxed{\underline{\phi}_t \cdot \underline{\underline{I}}^{n+1}(\underline{\phi}^{\dagger}_{xx}) = d_t T_n + d_x \bar{X}_n \ ,} \tag{6.1.12}$$

and the *integro-differential* part

$$\boxed{\underline{\phi}_t \cdot \underline{\underline{I}}^{n}(\underline{\phi}^{\dagger}_{xt}) = d_x X'_n \quad .} \tag{6.1.13}$$

The following lemma, analagous to the commutation lemmas in previous chapters, will be useful in proving equation (6.1.10).

Lemma (6.1): That

$$\boxed{\underline{f} \cdot \underline{\underline{I}}^{n}(\underline{g}_x) \triangleq - \underline{g}^{\dagger} \cdot \underline{\underline{I}}^{n}(\underline{f}^{\dagger}_x) \quad ,} \tag{6.1.14}$$

provided that either

$$\underline{\phi}_x \cdot \underline{\underline{I}}^{i}(\underline{f}^{\dagger}_x) \approx d_x R^i \quad ,$$

or (6.1.15)

$$(0,1) \cdot \underline{\underline{I}}^{n-i-1}(\underline{g}_x) \approx d_x S^i \quad ,$$

for all $i < n$, where R^i and S^i are acceptable fluxes. [Recall that $\triangleq$ means *equals within an x-derivative,* and $\approx$ means *is equivalent to,* in the sense of appendix D].

Proof: the left-hand side of equation (6.1.14) is

$$\underline{f}\cdot\underline{\underline{I}}^n(\underline{g}_x) = \underline{f}\cdot\begin{pmatrix} \frac{1}{2}\phi_{1x} & , & \phi_{2x} + \frac{1}{2}\phi_{2xx}\int \\ 0 & , & \frac{1}{2}\phi_{1x} + \frac{1}{2}\phi_{1xx}\int \end{pmatrix}\underline{\underline{I}}^{n-1}(\underline{g}_x) \ , \qquad (6.1.16)$$

$$= \frac{d}{dx}\,\frac{1}{2}\,\{[0,\ \underline{\phi}_x\cdot\underline{f}^\dagger\int - \int\overline{\underline{\phi}_x\cdot\underline{f}_x^\dagger}\int\,]\ \underline{\underline{I}}^{n-1}(\underline{g}_x)\}$$

$$+ \int\ [\underline{\underline{I}}(\underline{f}_x^\dagger)]^\dagger\cdot\underline{\underline{I}}^{n-1}(\underline{g}_x) \quad . \qquad (6.1.17)$$

Repeated application of this result to the last term arising out of it gives

$$\underline{f}\cdot\underline{\underline{I}}^n(\underline{g}_x) = \frac{d}{dx}\,\frac{1}{2}\sum_{i=0}^{n-1}\{[0,\ \underline{\phi}_x\cdot\int\overline{\underline{\underline{I}}^i(\underline{f}_x^\dagger)}\int$$

$$- \int\overline{\underline{\phi}_x\cdot\underline{\underline{I}}^i(\underline{f}_x^\dagger)}\int]\underline{\underline{I}}^{n-i-1}(\underline{g}_x) + 2\underline{g}^\dagger\cdot\int\overline{\underline{\underline{I}}^n(\underline{f}_x^\dagger)}\}$$

$$- \underline{g}^\dagger\cdot\underline{\underline{I}}^n(\underline{f}_x^\dagger) \quad . \qquad (6.1.18)$$

This completes the proof of lemma (6.1).

(6.1.i) The Partial Differential Part

To prove equation (6.1.12), a Lagrangian will be derived for the equation

$$\underline{\underline{I}}^n(\underline{\phi}_{xx}^\dagger) = 0 \quad , \qquad (6.1.19)$$

using the theory of Atherton and Homsy (1975) for solving the inverse problem of the calculus of variations. This theory has been briefly presented in appendix J, for the case of one equation of motion in one scalar field variable. Atherton and Homsy make the natural extension to vector variables and several equations, which will be used with only brief explanations in this chapter.

A Lagrangian will exist for equation (6.1.19) if and only if the Frechet derivative of the left-hand side exists and is symmetrical. It will be shown that the left-hand side of equation (6.1.19) is a *polynomial* in $\underline{\phi}$, $\underline{\phi}_x$, and $\underline{\phi}_{xx}$, which will ensure the existence of the Frechet derivative.

Assume that

$$\underline{\underline{I}}^i(\underline{\phi}^{\dagger}_{xx}) = \underline{P}^i_x \quad , \tag{6.1.20}$$

for all $i < n$, where $\underline{P}^i$ is a polynomial in $\underline{\phi}$, $\underline{\phi}_x$, and $\underline{\phi}_{xx}$. Then

$$\underline{\underline{I}}^n(\underline{\phi}^{\dagger}_{xx}) = \underline{\underline{I}}(\underline{P}^{n-1}_x) \quad , \tag{6.1.21}$$

$$= \frac{d}{dx}\begin{pmatrix} \tfrac{1}{2}\phi_{2x} \; P_2^{n-1} \\ P_1^{n-1} + \tfrac{1}{2}\phi_{1x}P_2^{n-1} \end{pmatrix} + \tfrac{1}{2}\underline{\phi}_x \cdot \underline{P}^{n-1}_x \quad . \tag{6.1.22}$$

The last term in equation (6.1.22) can be dealt with by using lemma (6.1),

$$\tfrac{1}{2}\underline{\phi}_x \cdot \underline{P}_x^{n-1} = \tfrac{1}{2}\ \underline{\phi}_x \cdot \underline{\underline{I}}^{n-1}(\underline{\phi}_{xx}^{\dagger}) \quad , \tag{6.1.23}$$

$$= -\ \tfrac{1}{2}\underline{\phi}_x \cdot \underline{\underline{I}}^{n-1}(\underline{\phi}_{xx}^{\dagger}) + \frac{d}{dx}\ Q^{n-1} \quad , \tag{6.1.24}$$

where by reference to equation (6.1.18) it can be seen that all terms in Q^{n-1} are polynomials in $\underline{\phi}, \underline{\phi}_x$, and $\underline{\phi}_{xx}$, by virtue of assumption (6.1.20). Hence

$$\tfrac{1}{2}\underline{\phi}_x \cdot \underline{P}_x^{n-1} = \frac{d}{dx}\ \tfrac{1}{2}Q^{n-1} \quad , \tag{6.1.25}$$

and

$$\boxed{\underline{\underline{I}}^{n}(\underline{\phi}_{xx}^{\dagger}) = \underline{P}_x^{n} \quad ,} \tag{6.1.26}$$

where P^n is a polynomial in $\underline{\phi}, \underline{\phi}_x$, and $\underline{\phi}_{xx}$.

Since

$$\underline{\underline{I}}(\underline{\phi}_{xx}^{\dagger}) = \frac{d}{dx}\begin{pmatrix} \phi_{1x}\phi_{2x} \\ \phi_{2x} + \tfrac{1}{2}\phi_{1x}^2 \end{pmatrix} \quad , \tag{6.1.27}$$

the inductive process is started, and equation (6.1.26) is proved for all n. Hence the Frechet derivative of equation (6.1.19) exists, and is given by

$$\underline{\rho}\cdot[\underline{\underline{I}}^{n}(\underline{\phi}_{xx}^{\dagger})]'_{\underline{\phi}}(\underline{\sigma}) = \underline{\rho}\cdot(\underline{P}_x^{n})'_{\underline{\phi}}(\underline{\sigma}) \quad , \tag{6.1.28}$$

$$\equiv \underline{\rho}\cdot\begin{pmatrix} (P_{1x}^{n})'_{\phi_1}(\sigma_1) + (P_{1x}^{n})'_{\phi_2}(\sigma_2) \\ (P_{2x}^{n})'_{\phi_1}(\sigma_1) + (P_{2x}^{n})'_{\phi_2}(\sigma_2) \end{pmatrix} \quad . \tag{6.1.29}$$

[$\underline{\rho}$ and $\underline{\sigma}$ are arbitrary vector functions of $\underline{\phi}$]. The proof that the Frechet derivative is symmetrical is by induction. Assume that

$$\underline{\rho}\cdot(\underline{P}_x^i)'_{\underline{\phi}}\ (\underline{\sigma}) \triangleq \underline{\sigma}\cdot(\underline{P}_x^i)'_{\underline{\phi}}(\underline{\rho}) \tag{6.1.30}$$

for all $i < n$. Then

$$\underline{\rho}\cdot(\underline{P}_x^n)'_{\underline{\phi}}(\underline{\sigma}) = \underline{\rho}\cdot[\underline{\underline{I}}(\underline{P}_x^{n-1})]'_{\underline{\phi}}(\underline{\sigma}) \quad . \tag{6.1.31}$$

The right-hand side of equation (6.1.31) is

$$\underline{\rho}\cdot(\underline{\underline{I}})'_{\underline{\phi}}(\underline{\sigma})(\underline{P}_x^{n-1}) + \underline{\rho}\cdot\underline{\underline{I}}(\underline{P}_x^{n-1})'_{\underline{\phi}}(\underline{\sigma}) \ , \tag{6.1.32}$$

where

$$(\underline{\underline{I}})'_{\underline{\phi}}(\underline{\sigma}) \equiv \begin{pmatrix} \tfrac{1}{2}\sigma_{1x}, & \sigma_{2x} + \tfrac{1}{2}\sigma_{2xx}\int \\ 0\ , & \tfrac{1}{2}\sigma_{1x} + \tfrac{1}{2}\sigma_{1xx}\int \end{pmatrix} \quad . \tag{6.1.33}$$

Using lemma (6.1), the last term in expression (6.1.32) is

$$\underline{\rho}\cdot\underline{\underline{I}}(\underline{P}_x^{n-1})'_{\underline{\phi}}(\underline{\sigma}) \triangleq -\ [(\underline{P}^{n-1})'_{\underline{\phi}}(\underline{\sigma})]^{\dagger}\cdot\underline{\underline{I}}(\underline{\rho}_x^{\dagger}) \ , \tag{6.1.34}$$

$$\triangleq [\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)]\cdot[(\underline{P}_x^{n-1})'_{\underline{\phi}}(\underline{\sigma})] \quad . \tag{6.1.35}$$

Note that the second of requirements (6.1.15) is met in this case. Assumption (6.1.30) gives

$$[\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)]\cdot[(\underline{P}_x^{n-1})'_{\underline{\phi}}(\underline{\sigma})] \mathrel{\hat{=}} \underline{\sigma}\cdot(\underline{P}_x^{n-1})'_{\underline{\phi}}[\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)] \quad . \tag{6.1.36}$$

Using equations (6.1.35) and (6.1.36), equation (6.1.31) becomes

$$\underline{\rho}\cdot(\underline{P}_x^{n})'_{\underline{\phi}}(\underline{\sigma}) \mathrel{\hat{=}} \underline{\sigma}\cdot(\underline{P}_x^{n-1})'_{\underline{\phi}}[\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)] + \underline{\rho}\cdot(\underline{\underline{I}})'_{\underline{\phi}}(\underline{\sigma})(\underline{P}_x^{n-1}) . \tag{6.1.37}$$

Applying result (6.1.37) to the first term on the right-hand side of equation (6.1.37), that equation becomes

$$\underline{\rho}\cdot(\underline{P}_x^{n})'_{\underline{\phi}}(\underline{\sigma}) \mathrel{\hat{=}} [\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)]\cdot(\underline{P}_x^{n-2})'_{\underline{\phi}}[\int \underline{\underline{I}}^{\dagger}(\underline{\sigma}_x)] + \underline{\rho}\cdot(\underline{\underline{I}})'_{\phi}(\underline{\sigma})(\underline{P}_x^{n-1}) + \underline{\sigma}\cdot(\underline{\underline{I}})'_{\underline{\phi}}[\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)](\underline{P}_x^{n-2}) . \tag{6.1.38}$$

The first term on the right-hand side of equation (6.1.38) is symmetrical by assumption (6.1.30). The remaining terms may be shown to be symmetrical within an x-derivative by explicit calculation,

$$\underline{\rho}\cdot(\underline{\underline{I}})'_{\underline{\phi}}(\underline{\sigma})(\underline{P}_x^{n-1}) + \underline{\sigma}\cdot(\underline{\underline{I}})'_{\underline{\phi}}[\int \underline{\underline{I}}^{\dagger}(\underline{\rho}_x)](\underline{P}_x^{n-2})$$

$$= \underline{\rho}\cdot \begin{bmatrix} \tfrac{1}{4}\phi_{1x}\sigma_{1x} + \tfrac{1}{2}\sigma_{2x} , & \tfrac{1}{2}\phi_{2x}\sigma_{1x} + \tfrac{1}{2}\phi_{1x}\sigma_{2x} \\ & + (\tfrac{1}{4}\phi_{1xx}\sigma_{2x} + \tfrac{1}{4}\phi_{1x}\sigma_{2xx})\int \\ 0 , & \tfrac{1}{4}\phi_{1x}\sigma_{1x} + (\tfrac{1}{4}\phi_{1xx}\sigma_{1x} \\ & + \tfrac{1}{4}\phi_{1x}\sigma_{1xx})\int \end{bmatrix} (\underline{P}_x^{n-2})$$

$$+ \underline{\sigma}\cdot\begin{bmatrix} \tfrac14\phi_{1x}\rho_{1x} + \tfrac12\rho_{2x} \, , & \tfrac12\phi_{2x}\rho_{1x} + \tfrac12\phi_{1x}\rho_{2x} \\ & +(\tfrac14\phi_{1xx}\rho_{2x} + \tfrac14\phi_{1x}\rho_{2xx})\int \\ 0 \, , & \tfrac14\phi_{1x}\rho_{1x} + (\tfrac14\phi_{1xx}\rho_{1x} \\ & + \tfrac14\phi_{1x}\rho_{1xx})\int \end{bmatrix}(\underline{P}_x^{n-2})$$

$$-\tfrac12(\rho_{1x}\sigma_{2x} + \rho_{2x}\sigma_{1x})P_1^{n-2} - \tfrac12\sigma_{2x}\rho_{2x}P_2^{n-2}$$

$$+ \sigma_1\rho_1(\tfrac14\phi_{1xx}P_{1x}^{n-2} + \tfrac12\phi_{2xx}P_{2x}^{n-2} + \tfrac14\phi_{1xxx}P_2^{n-2} - \tfrac14\phi_{2xx}P_2^{n-2}$$

$$- \tfrac14\phi_{2x}P_{2x}^{n-2}) - \tfrac12\sigma_{1x}\rho_{1x}\phi_{2x}P_2^{n-2} + \tfrac14(\sigma_1\rho_{1x} + \rho_1\sigma_{1x})\phi_{2xx}P_2^{n-2}$$

$$+ \frac{d}{dx}\,[\tfrac12\sigma_2\rho_{2x}P_2^{n-2} + \tfrac12\rho_1\sigma_{2x}P_1^{n-2} + \tfrac12\rho_2\sigma_{1x}P_1^{n-2} + \tfrac14\sigma_2\rho_1\phi_{1xx}P_2^{n-2}$$

$$+ \tfrac12\sigma_1\rho_{1x}\phi_{2x}P_2^{n-2}] \quad . \tag{6.1.39}$$

Hence assumption (6.1.30) implies that

$$\boxed{\underline{\rho}\cdot(\underline{P}_x^n)'_{\underline{\phi}}(\underline{\sigma}) \doteq \underline{\sigma}\cdot(\underline{P}_x^n)'_{\underline{\phi}}(\underline{\rho}) \quad .} \tag{6.1.40}$$

Since

$$\underline{\rho}\cdot(\underline{P}_x^1)'_{\underline{\phi}}(\underline{\sigma}) = \frac{d}{dx}(\rho_1\sigma_{1x}\phi_{2x} + \rho_1\sigma_{2x}\phi_{1x}$$

$$+ \rho_2\phi_{1x}\sigma_{1x} + \rho_2\sigma_{2x}) - \rho_{1x}\sigma_{1x}\phi_{2x} - (\rho_{1x}\sigma_{2x} + \rho_{2x}\sigma_{1x})\phi_{1x}$$

$$- \rho_{2x}\sigma_{2x} \, , \tag{6.1.41}$$

$$\hat{=}\ \underline{\sigma}\cdot(\underline{P}_x^1)'(\underline{\rho}) \quad , \tag{6.1.42}$$

the induction is started, and equation (6.1.40) is proved for all *n*.

Since the Frechet derivative is symmetrical, equation (6.1.19)

$$\underline{\underline{I}}^n(\underline{\phi}_{xx}^\dagger) = 0 \tag{6.1.43}$$

has a Lagrangian given by [see Atherton and Homsy (1975)]

$$\boxed{L_n = \underline{\phi}\cdot\int_0^1 \underline{P}_x^n(\lambda\underline{\phi})\,d\lambda \quad .} \tag{6.1.44}$$

Clearly, L_n will be a polynomial in $\underline{\phi}, \underline{\phi}_x$, and $\underline{\phi}_{xx}$, and each L_n will be linearly independent of the other.

Since L_n has no explicit time-dependence, energy is conserved [see appendix C], and Noether's relation (2.0.4) gives

$$\boxed{\underline{\phi}_t\cdot\underline{\underline{I}}^n(\underline{\phi}_{xx}^\dagger) = \frac{d}{dt}\,L_n + \frac{d}{dx}[-\underline{\Pi}^x(L_n)\cdot\underline{\phi}_t] \quad ,} \tag{6.1.45}$$

where

$$\underline{\Pi}^x \equiv \sum_{a=0}^{\infty}\sum_{b=0}^{a}(-1)^b\, d_{\mu_1}\cdots d_{\mu_b}\left(\frac{\partial}{\partial\underline{\phi}_{x\mu_1\cdots\mu_a}}\right) d_{\mu_{b+1}}\cdots d_{\mu_a} \ , \tag{6.1.46}$$

a natural extension of the definition (2.0.6) of Π^μ.

This completes the proof of equation (6.1.12), with

$$T_n = L_{n+1} \quad ,$$

$$\bar{X}_n = - \underline{\Pi}^x(L^n)\cdot\underline{\phi}_t \quad . \tag{6.1.47}$$

(6.1.ii) <u>The Integro-Differential Part</u>

It remains to prove equation (6.1.13),

$$\boxed{\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xt}) = \frac{d}{dx}\, X_n' \quad .} \tag{6.1.48}$$

Lemma (6.1) can be used to give

$$\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xt}) \triangleq - \underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xt}) \quad , \tag{6.1.49}$$

since the second of requirements (6.1.15) is met as follows,

$$(0,1)\cdot\underline{\underline{I}}^i(\underline{\phi}^\dagger_{xt}) = (0,1)\cdot\underline{\underline{I}}^i[\underline{\phi}^\dagger_{xt} + \underline{\underline{I}}(\underline{\phi}^\dagger_{xx})]$$

$$-(0,1)\cdot\underline{\underline{I}}^{i+1}(\underline{\phi}^\dagger_{xx}) \quad , \tag{6.1.50}$$

$$\approx -(0.1)\cdot\underline{P}_x^{i+1} \quad , \tag{6.1.51}$$

since the first term on the right of equation (6.1.50) is zero for solutions to

$$\underline{\underline{I}}^k[\underline{\phi}^\dagger_{xt} + \underline{\underline{I}}(\underline{\phi}^\dagger_{xx})] = 0 \quad , \quad k < i+1 \quad . \tag{6.1.52}$$

Hence equation (6.1.49) holds, and can be written

$$\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{\phi}^{\dagger}_{xt}) = d_x X_n' \quad , \tag{6.1.53}$$

where X_n' is an acceptable flux. This completes the proof of the integro-differential part.

Adding equations (6.1.12) and (6.1.13) gives equation (6.1.10),

$$\boxed{\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{F}) = d_t T_n + d_x X_n \quad ,} \tag{6.1.54}$$

with

$$T_n = L_n = \underline{\phi} \cdot \int_0^1 \underline{P}^n_x(\lambda\underline{\phi})\,d\lambda \quad , \tag{6.1.55}$$

$$X_n = X_n' + \bar{X}_n \quad .$$

Since equation (6.1.54) has been proved to hold for all n, there is an infinite set of polynomial conserved *energy* densities L_n for each of the equations of motion

$$\underline{\underline{I}}^n(\underline{F}) = 0 \quad , \qquad n = 0, 1, \ldots , \tag{6.1.56}$$

and in particular for the classical nonlinear shallow-water equations

$$\underline{F} = 0 \ . \tag{6.1.57}$$

A comparison of the first four polynomial conserved *energy* densities with the densities obtained in appendix L from Benney's (1973) results, shows that they are equivalent within a constant factor [see appendix L]. Hence it is likely that every polynomial conserved density obtained from Benney's (1973) work can be identified as an energy density.

(6.2) AN INFINITE SET OF CONSERVED MOMENTUM DENSITIES FOR THE CLASSICAL SHALLOW-WATER EQUATIONS

It will be shown that the polynomial conserved densities L_n can be alternatively identified as momentum densities of the higher-order equations (6.1.56). This will be done by proving that

$$\boxed{\underline{\phi}_x \cdot \underline{\underline{I}}^n(\underline{F}) = d_t(-L_n) + d_x Y_n \quad .} \tag{6.2.1}$$

Equation (6.2.1) associates the transformation

$$\bar{\delta}\underline{\phi} = - \varepsilon\underline{\phi}_x \tag{6.2.2}$$

with the conservation law

$$d_t(-L_n) + d_x Y_n = 0 \quad , \tag{6.2.3}$$

which holds for solutions to the equations

$$\underline{\underline{I}}^i(\underline{F}) = 0 \quad , \quad i \leqslant n \; . \tag{6.2.4}$$

The *integro-differential* part of equation (6.2.1) is

$$\boxed{\underline{\phi}_x \cdot \underline{\underline{I}}^n(\underline{\phi}^{\dagger}_{xt}) = d_t(-L_n) + d_x \bar{Y}_n \quad .} \tag{6.2.5}$$

Equation (6.2.5) will be proved by applying lemma (6.1) to the left-hand side,

$$\underline{\phi}_x \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xt}) \doteq -\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xx}) \quad . \tag{6.2.6}$$

Requirement (6.1.15) is met by virtue of equation (6.1.25),

$$\underline{\phi}_x \cdot \underline{\underline{I}}^k(\underline{\phi}^\dagger_{xx}) = d_x Q^k \quad . \tag{6.2.7}$$

Recall equation (6.1.45),

$$\underline{\phi}_t \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xx}) = d_t L_n + d_x[-\underline{\Pi}^x(L_n)\cdot\underline{\phi}_t] \quad . \tag{6.2.8}$$

This implies that equation (6.2.6) can be written

$$\underline{\phi}_x \cdot \underline{\underline{I}}^n(\underline{\phi}^\dagger_{xx}) = d_t(-L_n) + d_x[-\underline{\Pi}^x(L_n)\cdot\underline{\phi}_t + W^n] \quad , \tag{6.2.9}$$

where W^n is some acceptable flux. This completes the proof of equation (6.2.5).

The partial differential part of equation (6.2.1) is

$$\boxed{\underline{\phi}_x \cdot \underline{\underline{I}}^{n+1}(\underline{\phi}^\dagger_{xx}) = d_x(Y_n - \bar{Y}_n) \quad .} \tag{6.2.10}$$

Equation (6.2.10) has already been proved to hold, and is presented at equation (6.2.7). Adding together equations (6.2.5) and (6.2.10) yields equation (6.2.1). *Hence the conserved densities* $-L_n$ *are alternatively identified as momentum densities of the higher-order classical shallow-water equations.*

CHAPTER VII

CONCLUSIONS

The first part of this thesis, up to and including section (3.6), is review material, presenting the background information which it was felt was needed for the following sections. The remainder of the thesis presents original material, with the exception of section (4.2).

In the original material, Noether's theorem has been used to *identify* the infinite sets of polynomial conserved densities found for certain nonlinear evolution equations, as energy or momentum densities of higher-order equations whose solution sets contain that of the evolution equation under consideration. Noether's theorem enables this identification by associating each polynomial conserved density with an infinitesimal time or space translation on each higher-order enveloping equation.

Each nonlinear evolution equation considered in previous chapters [with the exception of the Sine-Gordon equation, a somewhat special case], can be written in the form [using scalars for simplicity]

$$\boxed{F \equiv \phi_{xt} + O(\phi_{xx}) = 0 \quad ,} \tag{7.0.1}$$

where O is a nonlinear integro-differential operator. The higher-order equations were then given by

$$O^n(F) = 0 \ , \quad n = 1,2, \ldots \ , \tag{7.0.2}$$

and the Noether's relation required to identify T_n as a conserved energy or momentum density for equation (7.0.2) was

$$\boxed{\phi_\mu O^n(F) = d_t T_n^\mu + d_x X_n^\mu \quad ,} \tag{7.0.3}$$

where

$$\{x^\mu\} = \{t,x\} \quad . \tag{7.0.4}$$

In all cases the X_n contained some terms which were composed of two integral expressions multiplied together. These terms had to be shown to be equivalent to acceptable flux terms, that is, terms which vanish whenever the dependent variables vanish.

(7.1) THE LINEAR CASE

The technique used on nonlinear evolution equations in this thesis can be shown to apply to *linear* equations possessing conserved energy or momentum densities, and not explicitly dependent on space or time. This feature was first mentioned in the Introduction, and will here be proved using generalised Noether's theorem. Let the linear equation be

$$\bar{F}(\phi,\phi_\mu, \ldots) = 0 \quad . \tag{7.1.1}$$

The energy or momentum conservation relation is

$$\phi_\mu \bar{F} = d_t T^\mu(\phi,\phi_\nu, \ldots) + d_x X^\mu(\phi,\phi_\nu, \ldots) . \tag{7.1.2}$$

The work in appendix H gives an infinity of conservation relations arising from equation (7.1.2),

$$\phi_{\mu n} d_x^n \bar{F} = d_t T_n^\mu + d_x X_n^\mu \quad , \tag{7.1.3}$$

where

$$\phi_{\mu n} \equiv d_x^n(\phi_\mu) , \; T_n^\mu \equiv T^\mu(\phi_n,\phi_{\nu n}, \ldots) \quad , \tag{7.1.4}$$

and taking for simplicity

$$\{x^\mu\} = \{x^0,x^1\} = \{t,x\} \quad . \tag{7.1.5}$$

Equation (7.1.3) can be rearranged to get

$$\boxed{\phi_\mu d_x^{2n}(\bar{F}) = d_t T_n^\mu + d_x [X_n^\mu + \sum_{i=0}^{n-1} \phi_{\mu i} \, d_x^{2n-i-1}(\bar{F})] .} \tag{7.1.6}$$

Comparison of equation (7.1.6) with equation (7.1.2) shows that T_n^μ may be regarded as an energy or momentum density for the higher-order linear equation

$$(d_x^2)^n(\bar{F}) = 0 \quad . \tag{7.1.7}$$

Note in particular that the operator d_x^2 in equation (7.1.7) is the *linearised* version of every nonlinear operator O used in previous chapters, with the exception of operator $\underline{\underline{I}}$ [chapter 6].

Hence the technique that has been applied to nonlinear equations in this thesis can be viewed as a generalisation of what happens with linear equations.

The part played by energy and momentum in this technique is suggestive of Fairlie's (1965) approach to the infinite number of conservation of *zilch* equations discovered by Kibble (1965) for the electromagnetic field. Fairlie obtains this infinite set by performing a Taylor expansion in a translation parameter on the energy-momentum tensor for the electromagnetic field.

(7.2) GENERALISATION

While the form that O would take for some arbitrary nonlinear equation is not obvious, certain general properties of O, commensurate with the properties of the nonlinear operators used in previous chapters, and sufficient for the existence of an infinity of polynomial conserved densities, will be discussed here.

The feature common to *all* nonlinear operators used in this thesis, and perhaps the most innovative feature of the technique used, is the *separation* of the problem into two problems, each handled in a different manner. One has been called the *partial differential* part, since it contains no integral terms. This part yields to analysis using the theory presented by Atherton and Homsy (1975) for deriving a Lagrangian for a nonlinear function. Subsequent application of Noether's theorem to the Lagrangian completes the proof of the partial differential part.

The other problem has been called the *integro-differential* part, since it contains integral terms. This part has yielded to an inductive approach, in which a *commutation relation* is proved, and the integral terms are shown to be acceptable.

In practice, it has not been difficult to find operators which satisfy one of these requirements, but the challenge has been to find an operator satisfying *both* requirements.

The breaking up of the problem (7.0.3) into two parts, one handled by a Lagrangian approach and the other by deriving a commutation relation, is presented in diagrammatic form for the energy and the momentum case in figure (7.1).

This division of the problem into two parts will be examined in more detail. In order to derive an infinite number of *energy* densities, the form of equation (7.0.3) would be

$$\boxed{\phi_t O^n(F) = d_t T_n^t + d_x X_x^t \quad .} \tag{7.2.1}$$

If the operator O is integro-differential in x, and if the equation of motion is of the form

$$F \equiv \phi_{xt} + K_x(\phi) = 0 \quad , \tag{7.2.2}$$

where K is some nonlinear function of ϕ and its x-derivatives, then the polynomials T_n^t [if they exist] can be required to be

FIGURE (7.1)

The Division into Integro-Differential and Partial Differential Parts

For an Infinity of Energy Densities

$\phi_t O^n(\phi_{xt})$	$+$	$\phi_t O^n[K_x(\phi)] = 0$
Integro-Differential (uses commutation relation)		Partial Differential (uses Lagrangian theory)

For an Infinity of Momentum Densities

$\phi_x O^n(\phi_{xt})$	$+$	$\phi_x O^n[K_x(\phi)] = 0$
Integro-Differential (uses commutation relation and Lagrangian theory)		Partial Differential (uses commutation relation)

polynomials in ϕ and its x-derivatives, as all t-derivatives can be removed by the substitution, from equation (7.2.2),

$$\phi_t = -K(\phi) \quad . \tag{7.2.3}$$

The left-hand side of equation (7.2.1) is divided into the two parts

$$\phi_t O^n(\phi_{xt}) + \phi_t O^n[K_x(\phi)] \quad . \tag{7.2.4}$$

It is clear that the first term in expression (7.2.4) cannot contribute to the time derivative of a polynomial in ϕ and its x-derivatives, so that under the earlier assumptions, the two parts must divide up as

$$\boxed{\begin{aligned} &\phi_t O^n(\phi_{xt}) = d_x \bar{X}_n^t \quad , && (7.2.5) \\ &\phi_t O^n[K_x(\phi)] = d_t T_n^t + d_x(X_n^t - \bar{X}_n^t) \quad . && (7.2.6) \end{aligned}}$$

Equation (7.2.5) will be integro-differential since operator O is integro-differential in x, and can be rewritten in the form of a *commutation relation*

$$\boxed{fO^{n+1}(g_x) \doteq -gO^{n+1}(f_x) \quad ,} \tag{7.2.7}$$

where equation (7.2.5) is satisfied if the commutation relation (7.2.7) holds for $f = g = \phi_t$. Recall that $\doteq$ means equals within an x-derivative.

In all of the cases considered so far, except the Sine-Gordon equation, the function K_x has taken the form

$$K_x(\phi) = O(\phi_{xx}) \quad , \tag{7.2.8}$$

so that equation (7.2.6) becomes

$$\boxed{\phi_t O^{n+1}(\phi_{xx}) = d_t T_n^t + d_x(X_n^t - \bar{X}_n^t) \quad .} \tag{7.2.9}$$

For the Sine-Gordon equation,

$$K_x(\phi) = \sin\phi \quad , \tag{7.2.10}$$

so that K is not analytic. However, this is a special case, as the operator S annihilates the sine term,

$$\phi_t S^n[K_x(\phi)] = \phi_t S^n(\sin\phi) = 0 \quad , \tag{7.2.11}$$

and in accordance with equations (7.2.11) and (7.2.6), the energy densities T_n^t are zero for the Sine-Gordon equation. It should be noted that even in this special case, an equation similar to equation (7.2.9) needed to be proved to obtain the infinite set of conserved momentum densities. This was equation (5.1.30),

$$\phi_{1t} G^n(\phi_x) = d_t T^n + d_x X'^n \quad . \tag{7.2.12}$$

Before considering the proof of equation (7.2.9), the derivation of an infinite number of *momentum* densities will be looked at. The appropriate form of equation (7.0.3) is

$$\boxed{\phi_x O^n(F) = d_t T_n^x + d_x X_n^x \quad .} \tag{7.2.13}$$

The same assumptions will be made as for the energy densities, that operator O is integro-differential in x and that equation (7.2.13) is divided into the two parts

$$\phi_x O^n(\phi_{xt}) + \phi_x O^n[K_x(\phi)] \quad . \tag{7.2.14}$$

Clearly, the second term in expression (7.2.14) cannot contribute to a time derivative of a polynomial in ϕ and its x-derivatives, so that equation (7.2.13) must divide up as

$$\phi_x O^n[K_x(\phi)] = d_x \bar{X}_n^x \quad , \tag{7.2.15}$$

$$\phi_x O^n(\phi_{xt}) = d_t T_n^x + d_t(X_n^x - \bar{X}_n^x) \quad . \tag{7.2.16}$$

If, as before, K_x is of the form

$$K_x(\phi) = O(\phi_{xx}), \tag{7.2.17}$$

equation (7.2.15) will be satisfied if the *commutation relation* (7.2.7),

$$\boxed{fO^{n+1}(g_x) \doteq - gO^{n+1}(f_x) \quad ,} \tag{7.2.18}$$

holds for $f = g = \phi_x$.

The remaining equations (7.2.9) and (7.2.16) have in practice yielded to a *Lagrangian* approach, whereby a Lagrangian is derived and Noether's theorem is applied to obtain a conservation relation. This approach has worked even though the left-hand side of equation (7.2.16) has been integro-differential in form. This is not surprising when it is realised that T_n^μ is a polynomial in all cases considered, and the integral terms appear only in the flux term [the x-derivative]. Once any integral terms on the left-hand side are safely tucked away into an x-derivative, the remaining partial differential part has been dealt with by deriving a Lagrangian. The theory for solving the inverse problem of the calculus of variations, presented by Atherton and Homsy (1975), has been applied to derive this Lagrangian in previous chapters. The derivation has depended on an inductive approach. This could be tidied up and simplified somewhat if the Atherton and Homsy results were extended to the case of one nonlinear integro-differential operator acting n times on some function, obtaining more specific conditions on the operator and the function for the nth case to have a Lagrangian. Such a Lagrangian may not exist in some cases, and a generalised Noether approach, which amounts to a head-on assault with an inductive proof, may be required.

The technique described in this section is *not* claimed to be exhaustive or complete. It merely summarises the common features of the approaches used on four particular problems in this thesis. Some of the possible variations of the technique [though by no means all] might include a different commutation relation, a simple example being

$$fO^n(g) \triangleq -gO^n(f) \quad . \tag{7.2.19}$$

This relation could prove the integro-differential part for the energy densities of an equation of the form

$$\phi_t + K(\phi) = 0 \quad . \tag{7.2.20}$$

Another possibility might be that the $\triangleq$ sign in the commutation relation (7.2.18) or (7.2.19) could mean equals within a derivative with respect to x and t. In this case the commutation relation may contribute integral terms to the conserved densities.

ACKNOWLEDGEMENTS

I am grateful to my supervisor, Professor W.L. Jones, for his continued guidance and support in this work, and for his invaluable insight and inspiration. I am also grateful to Professor B.G. Wybourne for his kind interest and assistance.

REFERENCES

Ablowitz, M.J., D.J. Kaup, A.C. Newell, H. Segur

(1973) *Nonlinear-Evolution Equations of Physical Significance,* Phys. Rev. Lett. 31, p. 125.

(1974) *Inverse Scattering Transform-Fourier Analysis for Nonlinear Problems,* Stud. Appl. Math. 53, p. 249.

Atherton, R.W. and G.M. Homsy

(1975) *On the Existence & Formulation of Variational Principles for Nonlinear Differential Equations,* Stud. Appl. Math. 54, p. 31.

Benney, D.J.

(1973) *Some Properties of Long Nonlinear Waves,* Stud. Appl. Math. 52, p. 45.

Boyer, T.H.

(1966) *Derivation of Conserved Quantities from Symmetries of the Lagrangian in Field Theory,* Am. J. Phys. 34, p. 475.

(1967) *Continuous Symmetries and Conserved Currents,* Ann. Phys. (N.Y.) 42, p. 445.

Candotti, E., C. Palmieri, B. Vitale

(1970) *On the Inversion of Noether's Theorem in the Lagrangian Formalism,* Nuovo Cim. 70A, p. 233.

(1972) *Universal Noether's Nature of Infinitesimal Transformations in Lorentz-Covariant Field Theories* Nuovo Cim. 7A, p. 271.

Dass, T.

(1966a) *Conservation Laws and Symmetries,* Phys. Rev. 145, p. 1011.

(1966b) *Conservation Laws and Symmetries, II.* Phys. Rev. 150, p. 1251.

Dothan, Y.

(1972) *Infinitesimal Invariance Transformations,* Nuovo Cim. 11A, p.499.

Eisenhart, L.P.

(1960) *"A Treatise on the Differential Geometry of Curves and Surfaces"* Reprinted, Dover (N.Y.), p.284.

Fairlie, D.B.

(1965) *Conservation Laws and Invariance Principles,* Nuovo Cim. 37 p. 897.

Fermi, E., J. Pasta & S. Ulam

(1955) Los Alamos Report LA 1940, [reproduced in Newell (1974)].

Gardner, C.S.

(1970) *Korteweg-de Vries Equation and Generalisations. IV. The Korteweg-de Vries Equation as a Hamiltonian System.* J. Math. Phys. 12, p. 1548.

Gardner, C.S., J.M. Greene, M.D. Kruskal & R.M. Miura

(1967) *Method for Solving the Korteweg-de Vries Equation,* Phys. Rev. Lett. 19, p. 1095.

(1974) *Korteweg-de Vries Equation and Generalisations. VI. Methods for Exact Solution,* Comm. P. App. Math. 27, p. 97.

Gardner, C.S., & G.K. Morikawa

(1960) New York Univ., Courant Inst. Math. Sci., Res. Rep. NYO-9082.

Gel'fand, I.M., & B.M. Levitan

(1955) *On the Determination of a Differential Equation from its Spectral Function,* Am. Math. Soc. Transl., Ser. 2. 1, p. 253.

Hill, E.L.

(1951) *Hamilton's Theorem and the Conservation Theorems of Mathematical Physics,* Rev. Mod. Phys. 23, p. 253.

Kay, I., & H.E. Moses

(1956) *Reflectionless Transmission through Dielectrics and Scattering Potentials,* J. Appl. Phys. 27, p. 1503.

Kibble, T.W.B.

(1965) *Conservation Laws for Free Fields,* J. Math. Phys. 6, p. 1022.

Korteweg, D.J., & G. de Vries

(1895) *On the Change of Form of Long Waves advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves,* Philos. Mag. 39, p. 422.

Kruskal, M.D.

(1974) *Nonlinear Wave Motion,* A.M.S. Lect. in App. Maths. 15, A.C. Newell ed., p. 67.

Kruskal, M.D., R.M. Miura, C.S. Gardner

(1970) *Korteweg-de Vries Equation and Generalisations. V. Uniqueness and Nonexistence of Polynomial Conservation Laws*, J. Math. Phys. 11, p 952.

Lamb, G.L. (Jr.)

(1970) *Higher conservation Laws in Ultrashort Optical Pulse Propagation*, Phys. Lett. 32A, p. 251.

Lax, P.D.

(1968) *Integrals of Nonlinear Equations of Evolution and Solitary Waves*, Comm. P. Ap. Maths, 21, p. 467.

(1976) *Almost Periodic Solutions of the KdV Equation*, SIAM Rev. 18, p. 351.

Leibovich, S.

(1970) *Weakly Nonlinear Waves in Rotating Fluids*, J. Fluid. Mech. 42, p. 803.

Leibovich, S., & A.R. Seebass

(1974) eds., *Nonlinear Waves*, Ithaca (N.Y.) Cornell Univ. Press.

Levinson, N.

(1953) *Certain Explicit Relationships between Phase Shift and Scattering Potential*, Phys. Rev. 89, p. 755.

Miura, R.M.

(1967) *Korteweg-de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation*, J. Math. Phys. 9, p. 1202.

Miura, R.M.

(1974) *Conservation Laws for the Fully Nonlinear Long Wave Equations*, Stud. Appl. Math. 53, p. 45.

(1976) *The Korteweg-de Vries Equation: A Survey of Results*, Siam Rev. 18, p. 412.

Miura, R.M., C.S. Gardner, M.D. Kruskal

(1968) *Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion*, J. Math. Phys. 9, p. 1204.

Nariboli, G.A., & Sedov, A.

(1970) *Burghers-Korteweg-de Vries Equation for Viscoelastic Rods and Plates,* J. Math. Anal. 32, p. 661.

Newell, A.C.

(1974) ed. *Nonlinear Wave Motion,* A.M.S. lect. in App. Maths. 15.

Noether, E.

(1918) *Invariante Variationsprobleme,* Kgl. Ges. d. Wiss. Nachrichten (Göttingen) Math. Phys. Klasse, p. 235. (In German).

Olver, P.J.

(1976) *Evolution equations possessing infinitely many symmetries,* J. Math. Phys. 18, p. 1212.

Palmieri, C., Vitale, B.

(1970) *On the Inversion of Noether's Theorem in the Lagrangian Formalism,* Nuovo Cim. 66A, p. 299.

Rosen, J.

(1971) *Conventional Formulations of Noether's Theorem,* Int. J. Theor. Phys. 4, p. 287.

(1972) *Noether's Theorem in Classical Field Theory,* Ann. Phys. (N.Y.) 69, p. 349.

(1974a) *Generalised Noether's Theorem I. Theory,* Ann. Phys. (N.Y.) 82, p. 54.

(1974b) *Generalised Noether's Theorem II. Applications,* Ann. Phys. (N.Y.) 82, p. 70.

Rund, H.

(1966) *The Hamilton-Jacobi Theory in the Calculus of Variations,* D. Van Nostrand Co. Ltd., p. 300.

Sagan, H.

(1969) *Introduction to the Calculus of Variations,* McGraw-Hill, p. 122.

Saletan, E.J., & A.H. Cromer

(1971) *Theoretical Mechanics,* J. Wiley and Sons Inc.

Sanuki, H., & K. Konno

(1974) *Conservation Laws of Sine Gordon Equation*, Phys. Lett. 48A, p. 221.

Schiff, L.I.

(1968) *Quantum Mechanics*, 3rd Ed., McGraw-Hill.

Schroder, U.E.

(1968) *Noether's Theorem and the Conservation Laws in Classical Field Theories*, Fortschr. d. Phys. 16, p. 357.

Scott, A.C., F. Chu, D.W. McLaughlin

(1973) *Solitons*, Proc. I.E.E.E. 61, p. 1443.

Steudel, H.

(1965) *Invariance Groups and Conservations Laws in Linear Field Theories*, Nuovo Cim. 39, p. 395.

(1975a) *Noether's Theorem and Higher Conservation Laws in Ultrashort Pulse Propagation*, Ann. Physik, 32, p. 205.

(1975b) *Noether's Theorem and the Conservation Laws of the Korteweg-de Vries Equation*, Ann. Physik. 32, p. 445.

(1975c) *Derivation of Continuous Set of Conservation Laws for the Modified Korteweg-de Vries Equation by Noether's Theorem*, Ann. Physik, 32, p. 459. (In German, abstract in English).

Su, C.S., & C.S. Gardner

(1969) *Korteweg-de Vries Equation and Generalizations. III. Derivation of the Korteweg-de Vries Equation and Burghers Equation*, J. Math. Phys. 10, p. 536.

Taniuti, T., & C.C. Wei

(1968) *Reductive Perturbation Method in Nonlinear Wave Propagation. I.*, J. Phys. Soc. Japan 24, p. 941.

Tappert, F., & C.M. Varma

(1970) *Asymptotic Theory of Self-Trapping of Heat Pulses in Solids*, Phys. Rev. Lett. 25, p. 1108.

Trautman, A.

(1967) *Noether Equations and Conservation Laws*, Comm. Math. Phys. 6, p. 248.

Wahlquist, H.D., & F.B. Estabrook

(1973) *Bäcklund Transformation for Solutions of the Korteweg-de Vries Equation*, Phys. Rev. Lett. 31, p. 1386.

(1975) *Prolongation Structures of Nonlinear Evolution Equations*, J. Math. Phys. 16, p. 1.

Wijngaarden, L. Van

(1968) *On the Equations of Motion for Mixtures of Liquid and Gas Bubbles*, J. Fluid. Mech. 33, p. 465.

Zabusky, N.J.

(1967) *Nonlinear Partial Differential Equations*, W. Ames ed., Academic Press, N.Y., p. 223.

(1973) *Solitons and Energy Transport in Nonlinear Lattices*, Comput. Phys. Commun. 5, p. 1.

Zabusky, N.J., & M.D. Kruskal

(1965) *Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States*, Phys. Rev. Lett. 15, p. 240.

Zielke, W.

(1974) *A Nonlinear System of Euler-Lagrange Equations. Reduction to the Korteweg-de Vries Equation and Periodic Solutions*, J. Math. Phys. 16, p. 1573.

APPENDIX A

In practical applications, the expansion of

$$\pi^{\nu}(L)\delta\phi = \sum_{a=0}^{\infty}\sum_{b=0}^{a} (-1)^b\, d_{\mu_1} \dots d_{\mu_b}\left(\frac{\partial L}{\partial\phi_{\mu_1 \dots \mu_a \nu}}\right) d_{\mu_{b+1}} \dots d_{\mu_a}\bar{\delta}\phi \tag{A.1}$$

in the equation (2.1.15):

$$\sum_a \frac{\partial L}{\partial\phi_{\mu_1 \dots \mu_a}}\,\bar{\delta}\phi_{\mu_1 \dots \mu_a} = d_{\mu}[\pi^{\mu}(L)\bar{\delta}\phi] + \bar{\delta}\phi E_{\phi}(L) \tag{A.2}$$

must be treated with caution if there are two or more independent variables x^{μ} and L contains second or higher derivatives of ϕ.

Even if it is realised that the sum on repeated indices $\{\mu_1 \dots \mu_a\}$ is over combinations only, it is easy to compute $\pi^{\nu}(L)\bar{\delta}\phi$ incorrectly. For example, let

$$L = L(\phi_{xxt}) \quad . \tag{A.3}$$

Then from the formula (A.1), it would appear that

$$\pi^{t}(L)\bar{\delta}\phi = \frac{\partial L}{\partial\phi_{xxt}}\,\bar{\delta}\phi_{xx} - d_x\left(\frac{\partial L}{\partial\phi_{xxt}}\right)\bar{\delta}\phi_x + d_x^2\left(\frac{\partial L}{\partial\phi_{xxt}}\right)\bar{\delta}\phi \ , \tag{A.4}$$

and

$$\pi^{x}(L)\bar{\delta}\phi = \frac{\partial L}{\partial\phi_{txx}}\,\bar{\delta}\phi_{tx} - d_x\left(\frac{\partial L}{\partial\phi_{txx}}\right)\bar{\delta}\phi_t + d_t d_x\left(\frac{\partial L}{\partial\phi_{txx}}\right)\bar{\delta}\phi \ . \tag{A.5}$$

However, this is incorrect. Working from the L.H.S. of (A.2), we can rederive $\pi^{\mu}(L)\bar{\delta}\phi$ for $L(\phi_{xxt})$, and see what it should correctly be:

$$\sum_a \frac{\partial L}{\partial\phi_{\mu_1 \cdots \mu_a}} \bar{\delta}\phi_{\mu_1 \cdots \mu_a} = \frac{\partial L}{\partial\phi_{xxt}} \bar{\delta}\phi_{xxt} \quad , \tag{A.6}$$

$$= d_t\left[\frac{\partial L}{\partial\phi_{xxt}} \bar{\delta}\phi_{xx}\right] - d_t\left(\frac{\partial L}{\partial\phi_{xxt}}\right)\bar{\delta}\phi_{xx} \quad , \tag{A.7}$$

$$= d_t\left[\frac{\partial L}{\partial\phi_{xxt}} \bar{\delta}\phi_{xx}\right] + d_x\left[- d_t\left(\frac{\partial L}{\partial\phi_{xxt}}\right)\bar{\delta}\phi_x\right.$$

$$\left. + d_x d_t\left(\frac{\partial L}{\partial\phi_{xxt}}\right)\bar{\delta}\phi\right] - \bar{\delta}\phi\; d_x^2 d_t\left(\frac{\partial L}{\partial\phi_{xxt}}\right) \quad , \tag{A.8}$$

or alternatively,

$$\frac{\partial L}{\partial\phi_{xxt}} \bar{\delta}\phi_{xx} = \frac{\partial L}{\partial\phi_{txx}} \bar{\delta}\phi_{txx} \quad ,$$

$$= d_x\left[\frac{\partial L}{\partial\phi_{txx}} \bar{\delta}\phi_{tx} - d_x\left(\frac{\partial L}{\partial\phi_{txx}}\right)\bar{\delta}\phi_t\right]$$

$$+ d_t\left[d_x^2\left(\frac{\partial L}{\partial\phi_{txx}}\right)\bar{\delta}\phi\right] - \bar{\delta}\phi\; d_x^2 d_t\left(\frac{\partial}{\partial\phi_{txx}}\right) \quad . \tag{A.9}$$

Noting that

$$\bar{\delta}\phi E_{\phi}[L(\phi_{xxt})] = - \bar{\delta}\phi d_x^2 d_t\left(\frac{\partial L}{\partial\phi_{xxt}}\right) \quad , \tag{A.10}$$

it is clear that calculation of $\pi^{\mu}(L)\bar{\delta}\phi$ from formula (A.1) has produced duplications and incorrect terms. Interpretation of formula (A.1) has to be done in the light of its derivation; the safest way of finding $\pi^{\mu}(L)\bar{\delta}\phi$ is to actually re-derive it in each application.

APPENDIX B

Conservation Laws Arising From Form-Invariance of Equation of Motion With Scalar Lagrangian Density

Consider any system with a Lagrangian which is independent of space and time except through the field variable:

$$L = L(\phi, \phi_{\mu}, \phi_{\mu\nu}, \ \ldots \ ) \quad . \tag{B.1}$$

Under an infinitesimal translation

$$\delta x^{\mu} = \varepsilon^{\mu} \quad , \tag{B.2}$$

the field variables vary as

$$\delta\phi \equiv \phi'(x') - \phi(x) = 0 \tag{B.3}$$

since the field variables at any point are unaltered. Hence from equation (2.1.11)

$$\bar{\delta}\phi = - \varepsilon^{\nu}\phi_{\nu} \quad . \tag{B.4}$$

The functional variation of the Lagrangian, using the property that it is a scalar density as in equation (2.3.1), is

$$\bar{\delta} L \equiv L'(\phi, \phi_\mu, \ldots) - L(\phi, \phi_\mu, \ldots) \tag{B.5}$$

$$= L'(\phi, \phi_\mu, \ldots) - L'(\phi', \phi'_\mu, \ldots) \quad , \tag{B.6}$$

which, using equation (B.3), becomes

$$\bar{\delta} L = L'(\phi, \phi_\mu \ldots) - L'(\phi, \phi_\mu \ldots) = 0 \quad . \tag{B.7}$$

That is, requirement (B) is satisfied, and since requirement (A) is satisfied by assumption, δJ is zero and hence the conservation law (2.3.6) holds:

$$d_\mu [\pi^\mu (L) \bar{\delta} \phi + L \delta x^\mu + G^\mu] = 0 \quad . \tag{B.8}$$

Substituting the values for $\bar{\delta}\phi$, δx^μ and G^μ, this becomes

$$d_\mu [\pi^\mu (L) \phi_\nu - L \delta^\mu_\nu] = 0 \quad , \tag{B.9}$$

which is the energy-momentum conservation law arising from translation - invariance of J for the system described by

$$E_\phi (L) = 0 \quad . \tag{B.10}$$

The energy-momentum tensor for a system is defined as

$$T^{\nu\mu} \equiv \pi^\mu (L) \phi_\nu - L \delta^\mu_\nu \quad . \tag{B.11}$$

The energy density is

$$T^{tt} \equiv \pi^{t}(L)\phi_{t} - L \qquad , \tag{B.12}$$

and its conservation arises from invariance of J under the time-translation, $\bar{\delta}\phi$ equals $- \varepsilon\phi_{t}$. The momentum density in the a direction

$$T^{at} \equiv \pi^{t}(L)\phi_{a} \tag{B.13}$$

and its conservation arises from invariance of J under translation in the a direction.

APPENDIX C

Condition on the Lagrangian for Conservation of Energy-Momentum Tensor.

Let $L = L[x^{\mu},\phi,\phi_{\mu}, \ldots]$. A condition will be derived that the system with the equation of motion

$$E_{\phi}(L) = 0 \tag{C.1}$$

conserves energy or momentum. Conservation of energy and momentum is traditionally associated with the invariance of the action integral under an infinitesimal translation, in space for momentum and in time for energy [see, for example, Saletan and Cromer (1971)]. Perform the infinitesimal translation

$$\delta x^{\mu} = \varepsilon^{\mu} \;, \qquad \delta\phi = 0 \tag{C.2}$$

on the system. Then recalling definition (2.5.2),

$$\delta L \equiv \sum_{a} \frac{\partial L}{\partial \phi_{\mu_1 \cdots \mu_a}} \delta\phi_{\mu_1 \cdots \mu_a} + \frac{\partial}{\partial x^{\mu}} (L \delta x^{\mu}) \;, \tag{C.3}$$

$$= \varepsilon^{\mu} \frac{\partial L}{\partial x^{\mu}} \;. \tag{C.4}$$

The necessary and sufficient condition (2.5.1) for invariance of the action integral is

$$\delta L = - d_{\mu} G^{\mu} \tag{C.5}$$

Hence conservation of energy or momentum requires

$$\varepsilon^{\mu} \frac{\partial}{\partial x^{\mu}} \quad L(x^{\mu},\phi,\phi_{\mu} \;\ldots) = - \frac{d}{dx^{\mu}} G^{\mu}(x,\phi,\phi_{\mu} \;\ldots\;) , \qquad (C.6)$$

where the derivative on the left-hand side is a partial derivative with respect to the explicit x-dependence of L, regarding ϕ and its derivatives as independent variables. Equation (C.6) requires for nontrivial L that

$$\frac{\partial L}{\partial x^{\mu}} = 0 \quad , \qquad (C.7)$$

that is, that L be independent of x^{μ}, except through the field variables and their derivatives.

If L *is not explicitly dependent on* t, *energy will be conserved. If* L *is not explicitly dependent on* x^a, *a space coordinate, then the* x^a *component of momentum will be conserved.* These general statements are an indication of the power of Noether's theorem as a general tool for predicting conservation laws.

APPENDIX D

CLASSIFICATION OF CONSERVED VECTORS

Steudel (1975a) and Rosen (1974a) have placed conserved vectors J^{μ} into different classes, according to the form their divergences take in the context of generalised Noether's theorem. Steudel's scheme is adopted here.

(1) FIRST KIND

A conserved vector J^{μ} of the *first kind* is one such that its divergence is linearly dependent on the equation of motion,

$$d_{\mu}J^{\mu} = \bar{\delta}\phi E_{\phi}(L) \quad . \tag{D.1}$$

That is, K is identically zero.

(2) SECOND KIND

A conserved vector J^{μ} of the *second kind* is one such that its divergence is zero for all solutions of the equation of motion, while there is no identity of the form of equation (D.1). That is, K is not identically zero.

(3) THIRD KIND

A conserved vector J^{μ} of the *third kind* is one such that its divergence is identically zero. Such a conservation law is called a strong conservation law, and is not a feature of the equation of motion of the system.

(4) FOURTH KIND

A conserved vector J^μ of the *fourth kind* is one for which the J^μ are themselves zero for solutions.

If a conserved vector is of the third or fourth kind, it is considered *trivial*. If two vectors are equal to each other within a trivially conserved vector, they are considered to be *equivalent*.

APPENDIX E

PROOF THAT THE EULER-LAGRANGE OPERATOR ON A DIVERGENCE IS IDENTICALLY ZERO

Recall the definition (2.0.5) of the Euler-Lagrange operator:

$$E_\phi(L) \equiv \sum_a (-1)^a d_{\mu_1} \ldots d_{\mu_a}\left(\frac{\partial L}{\partial \phi_{\mu_1 \ldots \mu_a}}\right) . \tag{E.1}$$

For simplicity of notation, the abbreviations

$$\phi_a \equiv \phi_{\mu_1 \ldots \mu_a} , \quad \phi_{a\nu} \equiv \phi_{\mu_1 \ldots \mu_a \nu} , \tag{E.2}$$

will be used in the remainder of this appendix.

The operator E_ϕ, operating on a divergence term

$$d_\nu G^\nu(x,\phi,\phi_\mu, \ldots) , \tag{E.3}$$

is

$$E_\phi(d_\nu G^\nu) = E_\phi(\sum_b \frac{\partial G^\nu}{\partial \phi_b} \phi_{b\nu} + \frac{\partial G^\nu}{\partial x^\nu}) , \tag{E.4}$$

recalling that d_ν is a total derivative, and ∂_ν is a partial derivative with respect to explicit x^ν-dependence only, regarding the ϕ_a as independent variables.

Using the definition of E_ϕ, equation (E.4) becomes

$$E_\phi(d_\nu G^\nu) = \sum_a (-1)^a d_{\mu_1} \ldots d_{\mu_a}\left[\sum_b \frac{\partial^2 G^\nu}{\partial \phi_a \partial \phi_b} \phi_{b\nu}\right.$$

$$+ \frac{\partial^2 G^\nu}{\partial\phi_a \partial x^\nu} + \sum_b \frac{\partial G^\nu}{\partial\phi_b} \delta^a_{b\nu} \Bigg] \quad , \tag{E.5}$$

$$= \sum_a (-1)^a \, d_{\mu_1} \cdots d_{\mu_a} \, d_\nu \left(\frac{\partial G^\nu}{\partial\phi_a} \right)$$

$$+ \sum_b (-1)^{b+1} \, d_{\mu_1} \cdots d_{\mu_b} \, d_\nu \left(\frac{\partial G^\nu}{\partial\phi_b} \right) \quad , \tag{E.6}$$

where the Dirac δ-function has been removed from the last term in equation (E.5) by replacing $\{\mu_1, \cdots, \mu_a\}$ with $\{\mu_1, \ldots, \mu_b, \nu\}$. Since b is a dummy index, being summed over, it can be replaced by a in equation (E.6) to get

$$E_\phi(d_\nu G^\nu) = \sum_a (-1)^a \, d_{\mu_1} \cdots d_{\mu_a} \, d_\nu \left(\frac{\partial G^\nu}{\partial\phi_a} \right)$$

$$+ \sum_a (-1)(-1)^a \, d_{\mu_1} \cdots d_{\mu_a} d_\nu \left(\frac{\partial G^\nu}{\partial\phi_a} \right) \quad , \tag{E.7}$$

and since the right-hand side is clearly zero,

$$E_\phi(d_\nu G^\nu) = 0 \quad . \tag{E.8}$$

APPENDIX F

INVERSE NOETHER'S THEOREM

The problem of inverse Noether's theorem is *whether a conservation law must have associated with it a transformation which leaves the action integral functionally invariant.* A nontrivial conservation law for any system with the equation of motion

$$F(x,\phi,\phi_{\mu} \ldots) = 0 \ , \tag{F.1}$$

must be able to be cast into generalised Noether form:

$$d_{\mu}J^{\mu} = - \bar{\delta}\phi F + K \ , \tag{F.2}$$

where K is zero for solutions and is linearly independent of F.

Recall the work of section (2.6) in which it was shown that the action integral is functionally invariant if and only if K is identically zero. This means that *the problem of inverse Noether's theorem is whether* K *must be identically zero for a conservation law.* This question has been answered in the affirmative by Dass (1966b) for all systems with equations of motion derivable from a Lagrangian, by Steudel [(1975), appendix B] for all systems with only one differential equation of motion, and by Candotti, Palmieri and Vitale (1970) for a large class of Lagrangian densities.

The strong converse of Noether's theorem as states by Steudel (1975) is that *for all systems with only one differential equation of motion, every conserved vector is equivalent to a conserved vector of the first kind,* that is, one for which K is identically zero. Hence, as Steudel states, there is a "biunique correspondence between Noether transformations and conservation laws," for these systems.

It will be proved here that *inverse Noether's theorem holds for any nontrivial conserved vector* J^μ *which is a differential expression involving* x, *the field variable* ϕ *and its derivatives.* The proof follows the same lines as that by Dass (1966b).

Let the equation of motion for the system be

$$F(x,\phi,\phi_\mu \cdots) = 0 \quad . \tag{F.3}$$

Then since the vector J^μ is conserved by virtue of the equation of motion,

$$d_\mu J^\mu \equiv f(F) \quad , \tag{F.4}$$

where f vanishes with its argument. Since J^μ is a differential expression, f must be of a form such that

$$d_\mu J^\mu \equiv (H + H^\mu d_\mu + H^{\mu\nu} d_\mu d_\nu + \ldots)F \tag{F.5}$$

where the H's are completely arbitrary functions of x,

ϕ and its derivatives. Use the following identity,

$$H^{\mu}d_{\mu}F = d_{\mu}(H^{\mu}F) - (d_{\mu}H^{\mu})F \quad , \tag{F.6}$$

to replace the second term on the right-hand side of (F.5), giving

$$d_{\mu}(J^{\mu}-H^{\mu}F) = (H - H^{\mu}_{\mu} + H^{\mu\nu}d_{\mu}d_{\nu} + \ldots) \ . \tag{F.7}$$

Note that this conserved vector is equivalent to J^{μ}, since a trivially conserved vector has been added. This process can be applied to the other derivative terms on the right-hand side of equation (F.7) to obtain

$$d_{\mu}[J^{\mu} + \sum_{a=1} \sum_{b=1}^{a} (-1)^{b} H^{\mu_1 \cdots \mu_a}_{\mu_1 \cdots \mu_{b-1}} F]$$
$$= \sum_{a=0} (-1)^{a} H^{\mu_1 \cdots \mu_a}_{\mu_1 \cdots \mu_a} F \ . \tag{F.8}$$

Since the summed terms in the divergence are themselves zero for solutions to equation (F.3), the whole vector is equivalent to J^{μ}. That is, J^{μ} is equivalent to a vector which is linearly dependent on the equation of motion, so that for this vector K is identically zero. Hence inverse Noether's theorem holds for this sort of conserved vector J^{μ}.

If K is not identically zero, then even though generalised Noether's theorem does associate a variation $\bar{\delta}\phi$ with the conservation law, this variation is not necessarily uniquely associated with the conservation law. Assume two conservation equations are associated with the same variation:

$$d_\mu J_1^\mu = \bar{\delta}\phi F + K_1 \quad , \tag{F.9}$$

$$d_\mu J_2^\mu = \bar{\delta}\phi F + K_2 \quad , \tag{F.10}$$

where the equation of motion is (F.3) above. Then

$$d_\mu (J_1^\mu - J_2^\mu) = K_1 - K_2 \quad , \tag{F.11}$$

that is, the difference between J_1^μ and J_2^μ is a conserved vector of the second kind, and is not necessarily trivial [see appendix D]. Hence the conserved vectors J_1^μ and J_2^μ are not necessarily equivalent.

APPENDIX G

CONSERVED DENSITIES AND FLUXES OF THE KDV EQUATION

Here are presented the first five conserved densities and fluxes of the KdV equation, from Miura, Gardner and Kruskal (1968). The conservation equations are

$$d_t T_n + d_x X_n = 0 , \qquad n = 1,2, \ldots, \tag{G.1}$$

and the densities and fluxes are [for equation (3.6.1)]:

$$T_1 = u . \tag{G.2}$$

$$X_1 = \frac{1}{2} u^2 + u_2 .$$

$$T_2 = \frac{1}{2} u^2 . \tag{G.3}$$

$$X_2 = \frac{1}{3} u^3 + uu_2 - \frac{1}{2} u_1^2 .$$

$$T_3 = \frac{1}{3} u^3 - u_1^2 . \tag{G.4}$$

$$X_3 = \frac{1}{4} u^4 + u^2 u_2 - 2uu_1^2 - 2u_1 u_3 + u_2^2 .$$

$$T_4 = \frac{1}{4} u^4 - 3uu_1^2 + \frac{9}{5} u_2^2 . \tag{G.5}$$

$$X_4 = \frac{1}{5} u^5 + u^3 u_2 - \frac{9}{2} u^2 u_1^2 + \frac{24}{5} u_0 u_2^2$$

$$- 6uu_1 u_3 + 3u_1^2 u_2 + \frac{18}{5} u_2 u_4 - \frac{9}{5} u_3^2 .$$

$$T_5 = \frac{1}{5} u^5 - 6u^2u_1^2 + \frac{36}{5} uu_2^2 - \frac{108}{35} u_3^2 \; . \tag{G.6}$$

$$X_5 = \frac{1}{6} u^6 + u^4u_2 - 8u^3u_1^2 + \frac{66}{5} u^2u_2^2 - 12u^2u_1u_3$$

$$+ 12uu_1^2u_2 - 3u_1^4 + \frac{72}{5} uu_2u_4 - \frac{72}{7} uu_3^2$$

$$- \frac{72}{5} u_1u_2u_3 + \frac{36}{35} u_2^3 - \frac{216}{35} u_3u_5 + \frac{108}{35} u_4^2 \; .$$

Note: the KdV equation is here in the form

$$u_t + uu_1 + u_3 = 0 \; . \tag{G.7}$$

APPENDIX H

INFINITE SETS OF CONSERVATION LAWS FOR LINEAR SYSTEMS

Let the linear equation of motion for a system be

$$F(\phi, \phi_\mu, \ldots) = 0 \quad . \tag{H.1}$$

Assume a conservation law exists for the system,

$$d_\mu J^\mu[\phi] = \bar{\delta}\phi[\phi] \,.\, F[\phi] \quad , \tag{H.2}$$

where

$$J^\mu[\phi] \equiv J^\mu(\phi, \phi_\mu, \ldots) \ , \tag{H.3}$$

etc.

Replace every ϕ in equation (H.2) with ϕ_n, where

$$\phi_n \equiv \phi_{\mu_1} \ldots {}_{\mu_n} \tag{H.4}$$

to get

$$d_\mu J^\mu[\phi_n] = \bar{\delta}\phi[\phi_n] \, F[\phi_n] \ . \tag{H.5}$$

Since the system is linear,

$$F[\phi_n] = d_{\mu_1} \ldots d_\mu \, F[\phi] \quad , \tag{H.6}$$

which is also zero for solutions to equation (H.1).

Hence

APPENDIX I

THE POLYNOMIAL CONSERVED DENSITIES AND FLUXES OF THE MODIFIED KdV EQUATION

Here are presented the first five conserved densities and fluxes of the modified KdV equation, from Miura, Gardner and Kruskal (1968). The conservation equations are

$$d_t T_n + d_x X_n = 0 \quad , \quad n = 1,2, \ldots \tag{I.1}$$

and the densities and fluxes are [for equation (4.1.2)]:

$$T_1 = \frac{1}{2} u^2 \quad .$$

$$X_1 = \frac{1}{4} u^4 + uu_2 - \frac{1}{2} u_1^2 \quad .$$

$$T_2 = \frac{1}{4} u^4 - \frac{3}{2} u_1^2 \quad .$$

$$X_2 = \frac{1}{6} u^6 + u^3 u_2 - 3u^2 u_1^2 - 3u_1 u_3 + \frac{3}{2} u_2^2 \quad .$$

$$T_3 = \frac{1}{6} u^6 - 5u^2 u_1^2 + 3u_2^2 \quad .$$

$$T_4 = \frac{1}{8} u^8 - \frac{21}{2} u^4 u_1^2 + \frac{63}{5} u^2 u_2^2 - \frac{63}{10} u_1^4 - \frac{27}{5} u_3^2 \quad .$$

$$\begin{aligned} T_5 = {} & \frac{1}{10} u^{10} - 18 u^6 u_1^2 + \frac{162}{5} u^4 u_2^2 - \frac{342}{5} u^2 u_1^4 \\ & - \frac{972}{35} u^2 u_3^2 + \frac{432}{7} u\, u_2^3 + \frac{5508}{35} u_1^2 u_2^2 \\ & + \frac{324}{35} u_4^2 \quad . \end{aligned}$$

For purposes of comparison, the conserved *energy* densities of the first three higher-order modified KdV equations have been determined, and written in terms of the conserved densities obtained by Miura, Gardner and Kruskal (1968).

$$L^1 = -\frac{1}{3} T_2 + d_x\left(\frac{1}{12}\phi\phi_1^3 + \frac{1}{2}\phi\phi_3 - \frac{1}{2}\phi_1\phi_2\right),$$

$$L^2 = -\frac{1}{6} T_3 + d_x\left(\frac{5}{12}\phi\phi_1\phi_2^2 + \frac{5}{12}\phi\phi_1^2\phi_3 + \frac{1}{36}\phi\phi_1^5\right.$$

$$\left. + \frac{1}{2}\phi\phi_5 - \frac{1}{2}\phi_1\phi_4 + \frac{1}{2}\phi_2\phi_3 - \frac{5}{12}\phi_1^3\phi_2\right),$$

$$L^3 = -\frac{5}{54} T_4 + d_x\left(\frac{5}{432}\phi\phi_1^7 + \frac{35}{108}\phi\phi_1^4\phi_3 + \frac{35}{54}\phi\phi_1^3\phi_2^2\right.$$

$$+ \frac{7}{12}\phi\phi_1^2\phi_5 + \frac{7}{3}\phi\phi_1\phi_2\phi_4 + \frac{7}{4}\phi\phi_1\phi_3^2 + \frac{35}{12}\phi\phi_2^2\phi_3 - \frac{35}{108}\phi_1^5\phi_2$$

$$- \frac{7}{12}\phi_1^3\phi_4 + \frac{1}{2}\phi\phi_7 - \frac{1}{2}\phi_1\phi_6 + \frac{1}{2}\phi_2\phi_5 - \frac{1}{2}\phi_3\phi_4 - \frac{7}{12}\phi_1^2\phi_2\phi_3$$

$$\left. - \frac{7}{12}\phi_1\phi_2^3\right) .$$

APPENDIX J

THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

The inverse problem of the calculus of variations amounts to the problem of finding a Lagrangian such that the Euler-Lagrange equation is the same as a given equation of motion. This appendix is a brief summary of some results obtained by Atherton and Homsy (1975) on this problem, as they apply to nonlinear partial differential equations.

An operator N on the space E is *potential* if there exists a functional F [mapping elements of E to the real line R] such that the *gradient* of F is N(u), for some $u \in E$.

The gradient of F is defined as the operator G(u) such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F(u + \varepsilon\phi) - F(u)] = \int G(u)\phi dV \quad , \qquad (J.1)$$

where the integral is over the physical domain. The left-hand side of equation (J.1) is simply the functional variation of F under the infinitesimal transformation

$$\delta u = \varepsilon\phi, \qquad \varepsilon \to 0 \quad , \qquad (J.2)$$

divided by ε.

The functional variation of the action integral

$$J = \int L(x, u, u_\mu, \ldots)\, dV \tag{J.3}$$

under the variation (J.2) is given by equation (2.1.19) as

$$\delta J = \int E_u(L)\, \varepsilon\phi dV \quad , \tag{J.4}$$

using the assumption that the field variables vanish on the boundary of integration. Equations (J.4) and (J.1) identify the gradient of J as

$$\text{grad } J = E_u(L) \quad . \tag{J.5}$$

Hence, if N is potential, there exists some functional

$$J = \int L dV \quad , \tag{J.6}$$

such that

$$E_u(L) = N(u) \quad . \tag{J.7}$$

that is, N(u) has a Lagrangian L.

Atherton and Homsy give the necessary and sufficient condition for N to be potential:

that the Frechet derivative of N *be symmetrical.*

An operation formula for the Frechet *differential* in the direction ϕ is

$$N_u'(\phi) = \lim_{\varepsilon \to 0} \frac{N(u + \varepsilon\phi) - N(u)}{\varepsilon} \quad , \tag{J.8}$$

$$= [\frac{\partial}{\partial \varepsilon} N(u + \varepsilon\phi)]_{\varepsilon=0} \quad . \tag{J.9}$$

If N is a differential operator,

$$N_u'(\phi) = \sum_{a=0} \frac{\partial N}{\partial u_{\mu_1 \cdots \mu_a}} \phi_{\mu_1 \cdots \mu_a} \quad , \tag{J.10}$$

The Frechet *derivative* is N_u'. The Frechet derivative is symmetrical if and only if

$$\int \psi N_u'(\phi)\, dV = \int \phi N_u'(\psi)\, dV \tag{J.11}$$

for any arbitrary $\psi, \phi \in E$. The integration is over the physical domain, so that the field variables vanish on the boundary.

If N is a potential operator, that is, if N has a symmetrical Frechet derivative, the corresponding Lagrangian is given by Atherton and Homsy as

$$L = u \int_0^1 N(\lambda u)\, d\lambda \quad , \tag{J.12}$$

where λ is a scalar.

APPENDIX K

THE POLYNOMIAL CONSERVED DENSITIES OF THE SINE-GORDON EQUATION

The Sine-Gordon equation describes the propagation of a plane monochromatic light pulse in a resonant two-level system in the slowly varying amplitude approximation [Steudel (1975a)]. Lamb (1970) gives a method based on a Hamiltonian approach for deriving an infinite number of polynomial conserved densities for this equation, and Sanuki and Konno (1974) derived a recurrence formula for these densities using the inverse spectral theory developed for solving certain nonlinear equations [see section (3)]. The Sine-Gordon equation used was

$$u_{xt} - \sin(u) = 0 \quad , \tag{K.1}$$

and the conservation laws are [using the notation of Sanuki and Konno, in which ϕ is some function of u and its derivatives]:

$$d_t\phi^{n+1} + d_x[- \frac{1}{4}\, \delta_0^n \cos(u) + \frac{1}{2}\, \phi^n \sin(u)] = 0 \quad , \tag{K.2}$$

where

$$\phi^n \equiv u_x\hat{\phi}^n \quad , \tag{K.3}$$

and

$$\hat{\phi}^{n+1} = \frac{1}{2}(-\hat{\phi}^n_x - \frac{1}{4}\, u_x\delta_0^n - u_x \sum_{i=0} \hat{\phi}^i\hat{\phi}^{n-i}) \quad . \tag{K.4}$$

The conserved densities ϕ^n are zero for even n, and polynomials for odd n. The first four nonzero densities obtained are (within a constant factor):

$$\phi^1 = u_1^2 \quad ,$$

$$\phi^3 = u_1u_3 + \frac{1}{4} u_1^4 \quad ,$$

$$\phi^5 = u_1u_5 + \frac{7}{4} u_1^3u_3 + \frac{11}{4} u_1^2u_2^2 + \frac{1}{8} u_1^6 \quad ,$$

$$\phi^7 = u_1u_7 + \frac{7}{2} u_1^2u_3^2 + \frac{19}{4} u_1u_2^2u_3 + \frac{3}{10} u_1^5u_3 + 2u_1^4u_2^2 + \frac{5}{64} u_1^8 \quad .$$

For comparison, the first three polynomial conserved *momentum* densities $(-L^n)$ are written here in terms of the variable u and the conserved densities ϕ^i above. The densities $(-L^n)$ are obtained from equation (5.1.54).

$$L^0 = \tfrac{1}{2}\phi^1 \quad ,$$

$$L^1 = \tfrac{1}{2}\phi^3 \quad ,$$

$$L^2 = \tfrac{1}{2}\phi^5 - \frac{d}{dx}\left(\frac{2}{8} u_1^3u_3\right) \quad .$$

APPENDIX L

THE CONSERVED DENSITIES OF THE CLASSICAL NONLINEAR SHALLOW-WATER EQUATIONS

Benney (1973) obtains an infinite set of conserved densities for the equations

$$u_x + v_y = 0 \quad , \tag{L.1}$$

$$u_t + uu_x + vu_y = -gh_x \quad , \tag{L.2}$$

$$v = 0, \quad y = 0 \tag{L.3}$$

$$h_t + uh_x - v = 0, \quad y = h \quad . \tag{L.4}$$

These equations reduce to the classical nonlinear shallow-water equations if the motion is irrotational, in which case

$$u = u(x,t), \quad v = -yu_x \quad . \tag{L.5}$$

Substitution of equations (L.5) into equations (L.1) to (L.4) yields equations (6.0.1),

$$\begin{aligned} h_t + uh_x + hu_x &= 0 \quad , \\ u_t + uu_x + gh_x &= 0 \quad . \end{aligned} \tag{L.6}$$

Equations (L.6) are transformed into the form used in chapter VI by

$$u = \phi_{1x} \ , \quad h = \phi_{2x} \ , \ g = 1 \quad . \tag{L.7}$$

Hence the infinite set of conserved densities for the classical nonlinear shallow-water equations can be obtained from Benney's set by the substitution,

$$u = \phi_{1x}(x,t), \quad v = -y\,\phi_{1xx}, \quad h = \phi_{2x}, \quad g = 1. \tag{L.8}$$

The first eight conserved densities obtained for the equations

$$\begin{aligned} &\phi_{2xt} + \phi_{1x}\phi_{2xx} + \phi_{1xx}\phi_{2x} = 0 \\ &\phi_{1xt} + \phi_{1x}\phi_{1xx} + \phi_{2xx} = 0 \end{aligned} \tag{L.9}$$

in this way are:

$$\begin{aligned} A_1 &= \phi_{2x}\,, \\ A_2 &= \phi_{1x}\phi_{2x}\,, \\ A_3 &= \tfrac{1}{2}\phi_{1x}^2\phi_{2x} + \tfrac{1}{2}\phi_{2x}^2\,, \\ A_4 &= \tfrac{1}{3}\phi_{1x}^3\phi_{2x} + \phi_{1x}\phi_{2x}^2\,, \\ A_5 &= \tfrac{1}{4}\phi_{1x}^4\phi_{2x} + \tfrac{3}{2}\phi_{1x}^2\phi_{2x}^2 + \tfrac{1}{2}\phi_{2x}^3\,, \\ A_6 &= \tfrac{1}{5}\phi_{1x}^5\phi_{2x} + 2\phi_{1x}^3\phi_{2x}^2 + 2\phi_{1x}\phi_{2x}^3\,, \\ A_7 &= \tfrac{1}{6}\phi_{1x}^6\phi_{2x} + \tfrac{5}{2}\phi_{1x}^4\phi_{2x}^2 + 5\phi_{1x}^2\phi_{2x}^3 + \tfrac{5}{6}\phi_{2x}^4\,, \\ A_8 &= \tfrac{1}{7}\phi_{1x}^7\phi_{2x} + 3\phi_{1x}^5\phi_{2x}^2 + 10\phi_{1x}^3\phi_{2x}^3 + 5\phi_{1x}\phi_{2x}^4\,. \end{aligned} \tag{L.10}$$

The first four conserved densities

$$T_n = \mathcal{L}_n \tag{L.11}$$

obtained as energy densities of the higher-order shallow-water equations are related to the densities A_i as follows,

$$T_0 = -A_3 + d_x(\tfrac{1}{3}\phi_1\phi_{1x}\phi_{2x} + \tfrac{1}{6}\phi_{1x}^2\phi_2 + \tfrac{1}{2}\phi_2\phi_{2x}) \; ,$$

$$T_1 = -\tfrac{3}{4}A_4 + d_x(\tfrac{3}{16}\phi_1\phi_{1x}^2\phi_{2x} + \tfrac{1}{4}\phi_1\phi_{2x}^2 + \tfrac{1}{16}\phi_2\phi_{1x}^3$$

$$+ \tfrac{1}{2}\phi_2\phi_{1x}\phi_{2x}) \; ,$$

$$T_2 = -\tfrac{1}{2} A_5 + d_x(\tfrac{1}{10}\phi_1\phi_{1x}^3\phi_{2x} + \tfrac{3}{8}\phi_1\phi_{1x}\phi_{2x}^2 + \tfrac{1}{40}\phi_2\phi_{1x}^4$$

$$+ \tfrac{3}{8}\phi_2\phi_{1x}^2\phi_{2x} + \tfrac{1}{4}\phi_2\phi_{2x}^2) \; ,$$

$$T_3 = -\tfrac{5}{16} A_6 + d_x(\tfrac{5}{96}\phi_1\phi_{1x}^4\phi_{2x} + \tfrac{3}{8}\phi_1\phi_{1x}^2\phi_{2x}^2 + \tfrac{5}{32}\phi_1\phi_{2x}^3$$

$$+ \tfrac{1}{96}\phi_2\phi_{1x}^5 + \tfrac{1}{4}\phi_2\phi_{1x}^3\phi_{2x} + \tfrac{15}{32}\phi_2\phi_{1x}\phi_{2x}^2) \; .$$